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MASTER THESIS

Higher Spin Wilson lines in AdS_3/ CFT_2

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Abstract

There have been recent developments in generalizing the Ryu-Takayanagi formula [1], which relates the area of a minimal surface in a d-dimensional Anti-de Sitter spacetime to the entanglement entropy of a $(d-1)$-dimensional CFT bounding such spacetime. One of such generalizations involves higher spin fields, relevant in many areas such as String Theory, although being notoriously hard to treat due to the difficulties arising when constructing their interactions.

In three dimensions there is a unique set of tools that allows for progress in this generalization program. In 2013, two equivalent proposals [2, 3] were developed, although only a semiclassical treatment was carried out. The purpose of this project is to develop a fully quantum mechanical treatment, based on very recent advances in doing so for the case of Einstein gravity [4].

We will begin reviewing basic notions of three dimensional gravity, from the tools that exploit its special features to the coupling to higher spin fields. We will subsequently motivate the introduction of Wilson lines in this setting and their evaluation, first in the case of Einstein gravity and then moving on to higher spin gravity. Finally, a quantum mechanical analysis of the higher spin Wilson line will be carried out.

Acknowledgements

First of all, I would like to thank Alejandra for her guidance and insights throughout the completion of this thesis. Thanks to her I have understood the beauty of three dimensional gravity and I have sharpened my knowledge about physics. I would also like to thank Jan for agreeing to be my second examiner.

Segundo, pero no menos importante, agradezco a mi familia el apoyo constante e incondicional que me han proporcionado durante estos años. A mis amigos en España, que tampoco me han dejado de apoyar en ningún momento y que siempre que vuelvo me hacen sentir como si nunca me hubiera ido.

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Finally I do really appreciate all the help coming from the PhDs and postdocs I have bothered over this year with my questions, such as Vassilis Anagiannis, Alex Belin, Marcos Crichigno, Victor Godet, Nabil Iqbal, Eva Llabrés, Greg Mathys, Beatrix Mühlmann...

Contents

Notation and conventions

Units: Throughout this thesis we will be using a unit system where all constants of nature except G are normalized to 1:

$$c = k_B = \hbar = 1$$

This makes equations much simpler and the results are the same once we recover the dimensionality with the appropriate combination of these constants. The choice of G not being normalized is just a convention carrying no special meaning, which is commonly used in the related literature.

Dimensions: The correspondence states that a conformal field theory in $d-1$ dimensions is dual to a gravitational theory in d. We will stick to this convention for each theory's dimension.

Metric signature: The metric will have the "mostly plus" signature convention:

$$\text{sign}(g) = (-, +, +, +, ...)$$

This is the standard notation in General Relativity references (with few exceptions), and it is the exact opposite to most of the particle physics literature.

Sometimes we will switch to Euclidean signature by means of a Wick rotation. This will be clear as we will use t for Lorentzian time, and τ for Euclidean, with $t = i\tau$

Indices: Spacetime indices will be denoted by Greek letters $\mu, \nu, ...$ while just spatial ones will be denoted by Latin letters $i, j, ...$. On the other hand, flat spacetime and gauge indices will be denoted by the first Latin letters $a, b, ...$

Einstein sum: Einstein summation convention (same indices up and down are implicitly summed over) will be used again and again:

$$\sum_i v_i w^i \equiv v_i w^i$$

The range of the index will be clear from the context, otherwise it will be specified.

Chapter 1

Introduction

I can't keep from fooling around with our irrefutable certainties. It is, for example, a pleasure knowingly to mix up two and three dimensionalities, flat and spatial, and to make fun of gravity.

M. C. Escher

Quantum gravity is famously hard. Several generations of physicists (including Einstein himself) tried to make sense of a quantum theory of gravity in accordance to both usual Quantum Field Theory and General Relativity. They all failed, however many lessons were learned along the way. During the last decades the way to tackle the problem has changed; instead of a direct approach researchers have been tackling different surrounding problems, from particular examples of other theories with similar breakdowns to toy models of quantum gravity where the setting is much more in control and understanding of the elements involved is greater.

The advent of the AdS/CFT correspondence was a huge stepping stone in this path. Since its initial discovery by Juan Maldacena in 1997 [5], it has provided a very useful setup where a gravity theory (Anti-de Sitter) is fully determined by a quantum theory (Conformal Field Theory). The tight relation between these two different theories is far-reaching, however there are many details that still need to be understood. The "AdS/CFT dictionary" is everything but complete, as more and more exotic theories are studied, each with its particular ingredients. Not only it is a fruitful resource for gravitational problems, but many features of Quantum Field Theories can also be studied with gravity lens[1] by making use of this dictionary. As such, this discovery has triggered a vast amount of research spanning several areas of physics and mathematics.

The initial relevance of this duality was in the context of the Information Paradox as it is the first and only working example of the Holographic Principle proposed, by Gerard 't Hooft as a resolution of said paradox. Within this paradigm, a quantity we all know from thermodynamics plays an utterly important role: entropy. In statistical mechanics it makes its appearance as a function counting some underlying degrees of freedom in the system. From the gravity picture, especially in black holes, the existence of such a quantity (so important that it received a name, the Bekenstein-Hawking entropy) has puzzled physicists for a long time. AdS/CFT is able to address this question, relying on the existence of degrees of freedom in the dual theory. Nevertheless, entropy is still an elusive quantity, and in the recent years one of its versions, entanglement entropy, has received special attention. This type of entropy is present in entangled systems, giving us a quantitative estimation of the information accessible of an entangled system.

[1]Metaphoric lens.

From the point of view of AdS, a dual description of the entanglement entropy in the CFT was missing until 2006, when Shinsei Ryu and Tadashi Takayanagi published their seminal work [1], relating the aforementioned quantity to the area of a minimal surface within the bulk gravitational theory. Since then, several attempts have been made to test it beyond simple examples and to generalize it to more exotic settings where boundary interpretations are still obscure. A particularly useful setup for all these matters is three dimensional gravity. It provides us with a surprisingly simple yet rich toy model where most higher dimensional features (existence of black holes, asymptotic symmetries, etc) can be studied without unnecessary complications. The expectation is for the results obtained to be able to shed light in more realistic problems. This is the case for both the Bekenstein-Hawking entropy and the Ryu-Takayanagi proposal: they admit simple expressions in three dimensions which could have been used to extrapolate their four dimensional analogs, had they been discovered earlier.

This project focuses on the so-called higher spin gravity. This subject deals with the interplay between gravity as we know it and higher spin fields, which are an extension to the Standard Model, arising naturally in String Theory as possible excitation modes of the string. Describing these particles and their interactions has proven to be a very complicated task, but successes in this direction come with the promise of a better understanding of String Theory, particularly its symmetries, as it has been conjectured to be a spontaneously broken phase of an underlying higher spin theory. In the context of higher spin gravity, three dimensions, once more, offer us a simple scenario where most of the complications alluded to previously will vanish, although the obscurity of their interpretation remains. Notably, they come equipped with a breakdown of usual geometric notions, rendering the metric tensor unreliable. Not all is lost, as three dimensional gravity brings enough machinery into play able to tackle this problem from a different (less geometrical and more topological) point of view. In particular, a natural topological observable, the Wilson line, has been proven the be the correct object to compute entanglement entropies [2, 3], both for pure (Einstein) and higher spin gravity. Not only that, but it has also been recently proven that these objects, in the pure gravity case, are particular cases of a propagator for a scalar field coupled to AdS_3 when the endpoints are attached to the boundary [4]. However, the same interpretation is lacking in the case of higher spin gravity. That is, to a certain extent, the end goal of this project.

The structure followed in this thesis has been chosen as a bottom-up approach to higher spin Wilson lines for pedagogical reasons. Chapter 2 will be devoted to an introduction to three dimensional gravity, from some solutions to their features. Chapter 3 will attempt to translate all we just discussed to Chern-Simons language. The reason for this will become clear in Chapter 4, where higher spin fields will be introduced, with special focus on their unique description available in three dimensions. Chapter 5 and Chapter 6 will review the use of Wilson lines as topological probes in three dimensional gravity. The former will introduce the main concepts and computational techniques in the context of pure gravity, whereas the latter will generalize them for higher spin gravity. During these two chapters we will be more explicit about contents and calculations, as we are no longer dealing with background material, but instead it will be the starting point for the results obtained during the completion of this thesis, which span the second part of the sixth chapter. We will finish with some discussion and outlook in Chapter 7.

Chapter 2

Metric formulation

The first step in this project is understanding Einstein gravity in three dimensions. In order to do so, in this chapter we will study it from the viewpoint of differential geometry. Our interest is twofold: to begin with, we need to introduce the reader to the special features of three dimensional gravity, namely its topological nature, how it allows for generalizations of four-dimensional concepts such as the existence black hole-like solutions, and the asymptotic behaviour of Anti-de Sitter (AdS), which will correspond to two copies of the Virasoro algebra in the Conformal Field Theory (CFT) defined at its (conformal) boundary. The latter was one of the first signs of a duality between an AdS space and a CFT at its boundary, although it took more than ten years for Maldacena to properly establish the relation [5].

Secondly, we want to pave the way for the next chapter, where all the content in this one will be rewritten by appealing to the topological properties of three dimensional gravity. For this reason we will put an emphasis on some computational routes over others, as they will be suitable for their posterior description.

A general background in General Relativity will be assumed, otherwise the reader is referred to e.g. [45]; although we will review the features needed as we go along. For a more in depth discussion of some of these topics, see [6].

2.1 General concepts

There are two important notions we need to discuss, the first one being three dimensional gravity, which, as we will see, is remarkably different from its higher dimensional counterparts. The other one is a solution to the Einstein's equations, Anti-de Sitter spacetime, whose properties allow for a description in terms of a dual field theory.

Throughout this first part we will comment on the several hints for the AdS/CFT correspondence, although we will not get into much detail for now. Later on we will devote some time for more explicit checks.

2.1.1 Three dimensional gravity

General Relativity has at its core the Einstein's equations:

$$G_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu}, \qquad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda. \tag{2.1}$$

Given a distribution of matter described by the energy-momentum tensor $T_{\mu\nu}$, these equations can be used to obtain the metric of the spacetime $g_{\mu\nu}$. However, this is not unique: different metrics can lead to the same Einstein tensor, hence same $T_{\mu\nu}$. Another way to see this is counting the (non-zero) independent components of the curvature tensors:

$\dim(\mathcal{M})$	Riemann	Ricci
n	$\frac{1}{12}n^2(n^2-1)$	$\frac{1}{2}n(n+1)$
3	6	6
4	20	10

Table 2.1: Number of independent non-zero components of the Riemann and Ricci tensors

As easily observed in Table 2.1, the Riemann tensor has more components than the Ricci from dimension 4 and above. However, Einstein's equations only make use of the Ricci tensor, which is the "diagonal part" of the Riemann tensor. The off-diagonal part of the Riemann, so important that it has been given a name, the Weil tensor $W^{\mu}{}_{\nu\rho\sigma}$, is not specified in the usual 4d General Relativity, i.e. there is no analog of the Einstein's equations fixing the Weil tensor. A very clear example of why this fact is relevant is considering vacuum solutions, $T_{\mu\nu} = 0$. In this setting there are different inequivalent solutions, such as flat space, gravitational waves or Schwarzschild. All those solutions differ in their respective Weyl tensor.

But this is not the case at hand. Three dimensional gravity is said to be a theory with no propagating degrees of freedom: Einstein's equations completely specify the Riemann tensor and the Weyl tensor vanishes identically. This has important consequences: consider the same example as before, the vacuum Einstein equations. Their solution is (almost) unique, hence there are no degrees of freedom that we need to specify as in the 4d case.

Why is it *almost* unique, then? First of all, the solution is unique up to diffeomorphisms. But something less trivial is that the above discussion is at the level of the metric, and the metric is a local object. This means that the metric is not aware of the topology of the space (a very simple example is the cylinder, whose metric is exactly that of 2d flat space, with a topological identification), hence one might find solutions with different topology, but same local information. We will see this come into play when we discuss the BTZ black hole.

2.1.2 Anti-de Sitter spacetime

Anti-de Sitter spacetime is a vacuum solution to the Einstein equations, with the presence of a negative cosmological constant, $\Lambda < 0$. It is named after Willem de Sitter, the first one to study this spacetime (along with the $\Lambda > 0$ case, dubbed de Sitter). In what follows, we will generally be more interested in describing this solution in terms of the AdS radius, ℓ, which in three dimensions is related to the cosmological constant by $\Lambda = -1/\ell^2$.

Both de Sitter and Anti-de Sitter spacetimes are maximally symmetric spaces, meaning that they have the same number of isometries as flat (Minkowski) spacetime, i.e. the same number of Killing vectors. While the Poincaré group $SO(1,3)$ is the symmetry group of four-dimensional Minkowski space, the symmetry group associated to the spacetime we are interested in, three-dimensional AdS, is $SO(2,2)$. Surprisingly enough, this is precisely the conformal group in two dimensions, a strong hint for the AdS/CFT duality. An utterly important feature of this group is the isomorphism $\mathrm{SO}(2,2) \simeq \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. We will explain the importance of this property in the next chapter, as it will be used throughout this thesis.

Coordinate patches

The most important coordinate patches of Anti-de Sitter spacetime are Global AdS_3, whose metric is given as

$$\mathrm{d}s^2_{\mathrm{Glob}} = -\left(\frac{r^2}{\ell^2}+1\right)\mathrm{d}t^2 + \left(\frac{r^2}{\ell^2}+1\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2, \tag{2.2}$$

and Poincaré AdS_3, with metric

$$\mathrm{d}s^2_{\mathrm{Poin}} = -\frac{r^2}{\ell}\mathrm{d}t^2 + \frac{\ell^2}{r^2}\mathrm{d}r^2 + \frac{r^2}{\ell^2}\mathrm{d}\phi^2. \tag{2.3}$$

where in both cases the coordinates take the usual range, $t \in \mathbb{R}$, $r \in \mathbb{R}^+$, $\phi \in [0, 2\pi]$. (another important patch is Rindler AdS_3 but we will not be interested in that case).

The relevance of these two patches is that they describe different AdS boundaries[1] at $r \to \infty$. Global AdS_3 has a boundary with the topology of a cylinder, whose non-compact direction is the timelike one. The Poincaré patch describes a subset of Global AdS_3 and has (polar) Minkowski spacetime as its boundary. This difference is important for the CFT description, as it will be the source of the distinction between the CFT defined on a plane or on a cylinder (although they are conformally related).

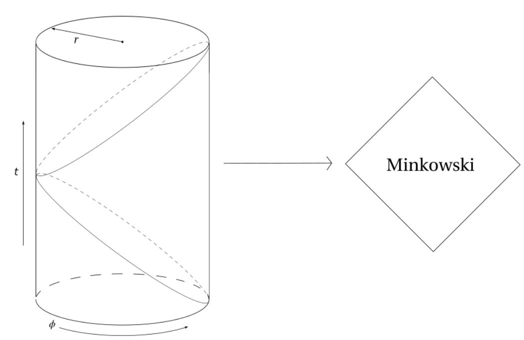

Figure 2.1: Sketch of Global AdS_3 with the Poincaré patch inside, where it can be seen that this patch asymptotes to a rhombus, which is the Penrose diagram of Minkowski spacetime.

We should stress an important characteristic of AdS, namely that its aforementioned boundary is timelike: it contains a time direction. This is something one does not find in, for instance, the conformal de Sitter boundary, and is one of the reasons why it is hard to realize holography in settings other than AdS, as it is not clear how to define a Quantum Field Theory without time.

[1] From now on, when we talk about the boundary of Anti-de Sitter spacetime it will be understood that it is a conformal boundary.

2.2 BTZ black hole

The above discussion on three dimensional gravity, and its lack of degrees of freedom, should raise some eyebrows if we begin to discuss black holes. In fact, it came as a surprise to the scientific community when a black hole solution in three dimensions was found. We will now discuss why this is not in conflict with the topological nature of three dimensional gravity and the properties of said solution.

The BTZ black hole [7], found by M. Bañados, C. Teittelboim and J. Zanelli, is a solution to the Einstein's equations in the vacuum with a negative cosmological constant. It is described by the metric

$$\mathrm{d}s^2_{\mathrm{BTZ}} = -N^2\mathrm{d}t^2 + N^{-2}\mathrm{d}r^2 + r^2(N^{\varphi}\mathrm{d}t + \mathrm{d}\varphi)^2, \tag{2.4}$$

where N and N^{ϕ} are, respectively, the lapse and shift functions, with the following form[2]

$$N^2(r) = -8MG + \frac{r^2}{\ell^2} + \frac{16G^2J^2}{r^2}, \quad N^{\varphi}(r) = -\frac{4GJ}{r^2}. \tag{2.5}$$

where M and J are, respectively, the mass and angular momentum of the black hole. These are the charges corresponding to the Killing vectors ∂_t and ∂_φ at infinity.

Before studying its properties, we need to address why is this even an allowed solution. The reason is that this is really Global AdS (2.2) in disguise. More precisely, this solution corresponds to a conical deficit or singularity in Global AdS_3, coming from a topological identification, which doesn't change the local structure seen by the metric. The easiest way to see that is considering the non-rotating solution ($J = 0$) and checking that it can be obtained by a topological identification in Global AdS_3 by $\varphi \sim \varphi + 2\pi\alpha$, $\alpha \in (0, 1)$, followed by a redefinition of the coordinates. The conical deficit parameter can be then shown to be related to the mass as $\alpha^2 = -8MG$. The rotating case is analogous, but including another identification in the time direction.

2.2.1 Structure and spectrum

The next task is understanding why it is a black hole solution. There are two geometrically characteristic features of a black hole in General Relativity:

1. **Singularity:** The $r = 0$ curvature singularity is not present in the BTZ case. The above reasoning for the BTZ being locally AdS implies that the black hole has constant negative curvature, everywhere including at $r = 0$. In fact, if one computes the Kretschmann invariant $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{12}{\ell^2}$, it is clear that this spacetime is not singular at the origin.

   This is no reason to dismiss this solution, as the singularity is not expected to be physical even in Schwarzschild: quantum gravity effects are supposed to take care of the singular behaviour.

2. **Event horizon:** This is the true distinguishing feature of a black hole, located where the metric is singular. For the BTZ this happens at radius

$$r_{\pm} = \ell\left[4MG\left(1 \pm \sqrt{1 - \left(\frac{J}{M\ell}\right)^2}\right)\right]^{1/2}. \tag{2.6}$$

[2] Note that I will be following the convention of [13] instead of the original ones.

The singularity located at r_+ is the null surface corresponding to an event horizon, as further inside the black hole, $r < r_+$ the time and radial coordinates exchange their roles, one would need to travel back in time (hence at a speed greater than the speed of light) in order to get out. We can draw its Penrose diagram, which for the $J = 0$ case resembles that of the Schwarzschild's black hole (Figure 2.2).

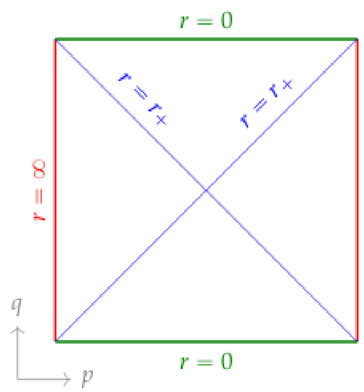

Figure 2.2: Penrose diagram for the non-rotating BTZ solution, where the non-singular behaviour at $r = 0$ has been made explicit. Figure from [13].

As in the four-dimensional counterpart, the event horizon will play a very important role when studying the thermodynamics of the black hole in the next subsection.

The rotating solution has an extra layer of complexity, as it shares with the Kerr black hole the existence of a surrounding ergosphere, which in this case is located at the vanishing of the g_{tt} component of the metric, $r_{\text{erg}} = \ell\sqrt{8MG}$. Observers at $r_+ < r_{\text{erg}}$ will then experience frame-dragging due to the black hole's rotation. The Penrose diagram for this rotating solution is also similar to that of the Kerr's black hole.

We can already discuss the spectrum of this black hole, that is, the allowed values of mass and angular momentum such that this solution is physically sound. Clearly, the existence of the event horizon r_+ imposes the conditions $|J| \leq M\ell \,\&\, M > 0$, coming from (2.6). The bound is saturated for the extremal black hole case, when the black hole's angular momentum is equal to its mass. The only stable solution for $M < 0$ (meaning that there is no conical singularity nor closed timelike curves) occurs for $J = 0$, $M = -1/8G$: we recover Global AdS_3. This was to be expected, as the argument relating the conical deficit angle with the mass implies that there is no conical singularity for $\alpha = 1$, which is precisely what we just found.

2.2.2 Thermodynamics

It is very well known that black holes are systems in thermal equilibrium, with the BTZ being no exception. There are several ways to compute the temperature associated to them; in the original paper [7] it is done via thermodynamical arguments relying on the Euclidean action, however in the present case it will be more instructive to discuss the so-called "smoothness in Euclidean signature" method.

To make things simple, let us discuss the procedure for the static ($J = 0$) BTZ case, although it can be generalized for the stationary case. The procedure goes as follows: first, we will omit the angular part of the metric (2.4), it will be of no importance here due to the rotational invariance.

Next, we expand the metric around the horizon, $r \to \epsilon + r_+$, $\epsilon \ll 1$, located at $r_+ = \sqrt{8MG}\ell$

$$\mathrm{d}s^2 = \left(8MG - \frac{r^2}{\ell^2}\right)\mathrm{d}t^2 + \left(-8MG + \frac{r^2}{\ell^2}\right)^{-1}\mathrm{d}r^2 \xrightarrow{r\to\epsilon+r_+} -\left(\frac{\cancel{\epsilon^2} + 2r_+\epsilon}{\ell^2}\right)\mathrm{d}t^2 + \left(\frac{\cancel{\epsilon^2} + 2r_+\epsilon}{\ell^2}\right)^{-1}\mathrm{d}\epsilon^2 \tag{2.7}$$

where we have dismissed the subleading terms. If we now change to Euclidean signature $t \to i\tau$ and redefine the radial coordinate as $\rho^2 = 2\ell^2\epsilon/r_+$, we end up with

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \frac{r_+^2}{\ell^4}\rho^2\mathrm{d}\tau^2. \tag{2.8}$$

Notice that this is nothing but the metric of a cone; as such the only way for it to be smooth (absent of a conical singularity) is by fixing the periodicity of τ such that the resulting metric is that of polar coordinates, $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\phi^2$, $\phi \in [0, 2\pi)$, as in that case the singularity is just a coordinate singularity rather than a geometrical one:

$$\rho^2\mathrm{d}\phi^2 = \frac{r_+^2}{\ell^4}\rho^2\mathrm{d}\tau^2 \Rightarrow \tau \in \left[0, \frac{2\pi\ell^2}{r_+}\right) \tag{2.9}$$

The inverse temperature β can be read off from the range of $\tau \in [0, \beta)$, therefore finding the temperature of the non-rotating BTZ black hole. For the rotating case this procedure also agrees with the thermodynamical computations of [7], yielding

$$T = \frac{1}{\beta} = \frac{r_+^2 - r_-^2}{2\pi\ell^2 r_+} \tag{2.10}$$

Another notoriously important thermodynamical quantity is the entropy, computed through usual thermodynamical arguments as

$$S = \frac{\pi r_+}{2G} = \frac{L_h}{4G}, \tag{2.11}$$

where L_h is the perimeter of the event horizon. This clearly coincides with the Bekenstein-Hawking entropy, $S_{\mathrm{BH}} = A/4G$, in the case of three spacetime dimensions.

2.3 Asymptotic Anti-de Sitter spacetimes

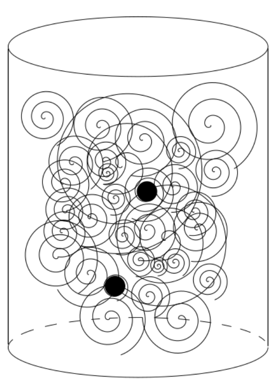

Figure 2.3: Rough AAdS depiction

Pure Anti-de Sitter spacetimes are well understood, one could even say that they are trivial, as there are not many interesting phenomena happening inside. What is more interesting in the context of AdS/CFT is considering a wider class of solutions, which contains pure AdS as a subset, but allows for more interesting effects and geometries to happen in the bulk while preserving the AdS-like boundary (or more concretely, its symmetries). These are the Asymptotically Anti-de Sitter (AAdS) spacetimes. In order to study them we need some kind of boundary condition, without being too restrictive, as we want diversity in the interior, such as black holes, galaxies, etc..

2.3.1 Fefferman-Graham expansion

The conditions we want to impose for a spacetime to be AAdS are quite obvious:

1. It is a solution to the Einstein's equations
2. It preserves the conformal structure at the boundary.

The way to implement these conditions in the metric was found by C. Fefferman and R. Graham [8], and it is then coined the Fefferman-Graham expansion :

$$\mathrm{d}s^2_{\mathrm{AAdS}} = \frac{\ell^2}{r^2}\mathrm{d}r^2 + g_{ij}(r,x^k)\,\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}\rho + (e^{2\rho/\ell}g^{(0)}_{ij} + g^{(2)}_{ij} + ...)\,\mathrm{d}x^i\mathrm{d}x^j \xrightarrow[r\to\infty]{} e^{2\rho/\ell}g^{(0)}_{ij}(x^k) + O(1), \tag{2.12}$$

where in this case the i, j, k indices also include time; the isolated coordinate is here $r = e^{\rho/\ell}$, which is the radial coordinate, hence the conformal boundary is located at $r, \rho \to \infty$. This has the following CFT interpretation: $g^{(0)}$ is to be identified with the metric of the CFT, which is fixed, and transformations (diffeomorphisms) leaving this conformal boundary invariant will act non-trivially on the subleading terms $g^{(2)}, g^{(4)}$....

Luckily enough, in three dimensional gravity the expansion is truncated at $g^{(4)}$, being an exact solution of Einstein's equations. Not only that, but also this last term $g^{(4)}$ is fixed in terms of $g^{(0)}$, $g^{(2)}$. This is not the only miracle, as the boundary conditions and equations of motion also imply that $g^{(2)}$ is traceless and divergence-free: just as the stress-tensor. Indeed, as we will see this interpretation will turn out to be correct.

Bañados found [9] the exact form of this expansion, in terms of the usual light-cone coordinates $x^\pm = t \pm \phi$ (and setting $\ell = 1$):

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 - 8\pi G\left(\mathcal{L}\,(\mathrm{d}x^+)^2 + \bar{\mathcal{L}}\,(\mathrm{d}x^-)^2\right) - \left(e^{2\rho} + 64\pi^2G^2\ell\,\mathcal{L}\,\bar{\mathcal{L}}\,e^{-2\rho}\right)\mathrm{d}x^+\mathrm{d}x^-\,, \tag{2.13}$$

where $\mathcal{L}(x^+), \bar{\mathcal{L}}(x^-)$ are unspecified functions (which, as we mentioned before, will correspond to the holomorphic and anti-holomorphic components of the stress tensor of the CFT) parameterizing the whole space of AAdS solutions. Spacetimes with different values of $\mathcal{L}(x^+), \bar{\mathcal{L}}(x^-)$ will be inequivalent, i.e. cannot be reached via diffeomorphisms.

Previously shown solutions can be recovered with different values of these functions:

- Global: $\mathcal{L} = \bar{\mathcal{L}} = 1/4\pi$.
- Poincare: $\mathcal{L} = \bar{\mathcal{L}} = 0$.
- BTZ: This black hole has an AAdS behaviour. In fact, constant and positive $\mathcal{L}, \bar{\mathcal{L}}$ corresponds to the black hole's mass and angular momentum by

$$\mathcal{L} = \frac{1}{4\pi}(M - J)\,, \qquad \bar{\mathcal{L}} = \frac{1}{4\pi}(M + J). \tag{2.14}$$

2.3.2 Asymptotic symmetries

One of the first hints of AdS/CFT was the discovery by J.D. Brown and M. Henneaux [10] of two copies of the Virasoro algebra in the boundary of AdS: if we study the "allowed" diffeomorphisms that leave said boundary invariant, the charges associated to their Killing vector satisfy two copies of the Virasoro algebra:

$$[L_n, L_m] = (n-m)\, L_{n+m} + \frac{c}{12} m\,(m^2-1)\, \delta_{m+n,0}, \quad c = \frac{3\ell}{4G}. \tag{2.15}$$

Since then, other easier ways to compute this central charge have been developed: for instance one can compute the boundary stress-tensor T^{ij} from the Gibbons-Hawking and counterterm contributions to the action [6],

$$T_{ij} = \frac{1}{8\pi G\ell}\left(g^{(2)}_{ij} - \mathrm{Tr}(g^{(2)})\, g^{(0)}_{ij}\right) \tag{2.16}$$

and its trace anomaly accounts for this central charge

$$\mathrm{Tr}(T) = -\frac{c}{24\pi} R. \tag{2.17}$$

Another (more interesting in what follows) way to obtain the value of the central charge is to consider infinitesimal diffeomorphisms ξ preserving the conformal boundary, that is, changing the metric $\mathscr{L} g = \delta g = \nabla_{(\mu}\xi_{\nu)}$ leaving $g^{(0)}$ invariant, but allowing for a change in the rest of the terms of the expansion. The set of these "allowed" transformations constitutes the so-called Asymptotic Symmetry Algebra. The most general of such transformations is

$$x^+ \longrightarrow x^+ + \epsilon(x^+) - \frac{\ell}{2} e^{-2\rho/\ell}\partial_-\bar{\epsilon}(x^-) \tag{2.18a}$$

$$x^- \longrightarrow x^- + \bar{\epsilon}(x^-) - \frac{\ell}{2} e^{-2\rho/\ell}\partial_+\epsilon(x^+) \tag{2.18b}$$

$$\rho \longrightarrow \rho - \frac{\ell}{2}\left(\partial_+\epsilon(x^+) + \partial_-\bar{\epsilon}(x^-)\right) \tag{2.18c}$$

where the functions $\epsilon, \bar{\epsilon}$ parameterize the infinitesimal transformation. This transformation in the metric induces a change on $\mathcal{L}$ in the following way

$$\mathcal{L} \longrightarrow \mathcal{L} + 2\partial_+\epsilon\mathcal{L} + \epsilon\partial_+\mathcal{L} - \frac{c}{24\pi}\partial_+^3\epsilon \quad c = \frac{3\ell}{4G}. \tag{2.19}$$

and analogously for $\bar{\mathcal{L}}$. But this none other than our familiar transformation law of the CFT stress-tensor, particularly (2.19) is the transformation of the holomorphic component, hence we can identify $\mathcal{L} \sim T_{++}$ and $\bar{\mathcal{L}} \sim T_{--}$. This is yet another check of the Asymptotic Symmetry Algebra of three dimensional AAdS spacetimes being Vir $\otimes$ Vir.

Chapter 3

Chern-Simons formulation

A very compelling reason as to why studying three dimensional gravity is its possibility to be written as a gauge theory, namely a Chern-Simons one. As we will see, diffeomorphisms (which are difficult to bypass in standard approaches to quantum gravity) become simple gauge transformations. Hence, this formulation is useful to work out lower dimensional analogs to full 4d gravitational problems, in a less convoluted fashion. In the present case, the motivation is different: it will allow us to introduce higher spin gravity in a fairly simple manner, as reviewed in the next chapter.

This formulation was first noticed by A. Achucarro and P. Townsend [11], although it was more thoroughly explored by E. Witten in his famous paper [12], where he explicitly showed how to construct such a gauge theory. In the following section we will show how it is done, not only to understand this construction but also to establish all the machinery we will be needing throughout the rest of this project. In this regard, the subsequent sections will be devoted to putting all said machinery into work by translating into a gauge theory language the notions introduced in the previous chapter.

Basic tools from the vielbein (frame) formulation of General Relativity will be employed, for the inexperienced reader we refer them to Appendix A for a short and self-contained introduction of the main features and techniques we will need. We will also frequently use the $SL(2,\mathbb{R})$ group and $\mathfrak{sl}(2,\mathbb{R})$ algebra, whose conventions can be found in Appendix B. For a more exhaustive review of the topics covered here, we refer to [13, 14].

3.1 Chern-Simons action for 3d gravity

Ever since the development of gauge theories in particle physics, physicists attempted to make sense of gravity as a gauge theory. General Relativity is invariant under diffeomorphisms, however if we want to translate this into a gauge theory, we need invariance under local (gauge) transformations, which diffeomorphisms are not.

In the frame formulation of gravity, in terms of vielbein $e_\mu{}^a$ and spin connection $\omega_\mu{}^a{}_b$, General Relativity is manifestly invariant under local gauge transformations, with gauge group $ISO(1, d-1)$, which is the d-dimensional Lorentz group + translations. If we introduce a cosmological constant $\Lambda \neq 0$, then in three dimensions the group is changed from $ISO(1,2)$ to either $SO(1,3)$ if $\Lambda > 0$ (corresponding to a de-Sitter space) or, if $\Lambda < 0$ then the group becomes the one corresponding to

Anti-de Sitter space, SO(2, 2), as we discussed in the previous chapter. From this point on we will focus on the latter case.

Action

One would want to think of the vielbein and spin connection as gauge fields for translations and rotations, respectively, which combined in some way would give rise to the Einstein-Hilbert action from a gauge theory Lagrangian. This can be done explicitly in three dimensions, as we shall now review.

First, we need to formulate the Einstein-Hilbert action in terms of frame fields. In this language, this action becomes

$$I_{\text{EH}}[g,\Gamma] = \frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^3x\sqrt{-g}(R-2\Lambda) = \frac{1}{16\pi G}\int_{\mathcal{M}} \epsilon_{abc}(e^a\wedge R^{bc} - \frac{\Lambda}{3}e^a\wedge e^b\wedge e^c), \tag{3.1}$$

where ϵ_{abc} is the Levi-Civita tensor. This action is sometimes referred to as the "tetradic Palatini action". In the usual Palatini action of General Relativity, the Christoffel symbols are regarded as separate variables to minimize independently from the metric and the variation with respect to them yields the torsion-free constraint. Here, in its tetradic version, we have a similar situation, a first order action where the variables to minimize are the vielbein and the spin connection and the resulting equations of motion are also Einstein's equations and the torsion-free condition, but this time in vielbein language.

As we have mentioned earlier, the idea is to map this action to a gauge one. We know that the Riemann tensor is expressed in terms of the spin connection, schematically as $R = \mathrm{d}\omega + \omega\wedge\omega$. This suggests that the action has the form $\int e\wedge(\mathrm{d}\omega + \omega\wedge\omega + e\wedge e)$, which seems somewhat similar to the Chern-Simons action with gauge group G:

$$I_{\text{CS}}[\mathcal{A}] = \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}(\mathcal{A}\wedge \mathrm{d}\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}), \tag{3.2}$$

where $\mathcal{A}$ is a $\mathfrak{g}$-valued connection, $\mathfrak{g}$ being the Lie algebra associated to the gauge group G. "Tr" we mean the non-degenerate quadratic bilinear on $\mathfrak{g}$: once we choose a basis in the Lie algebra $\mathcal{A} = \mathcal{A}^a T_a$, then the bilinear plays the role of the metric in the Lie algebra (the so-called Killing form), $d_{ab} = \mathrm{Tr}(T_a T_b)$.

Witten showed in his seminal paper [12] that both actions are equal up to boundary contributions, if we take the gauge field to be $\mathcal{A} = e^a P_a + \omega^a J_a \in \mathfrak{so}(2,2)$, where we have introduced the dual spin connection via $\epsilon^{abc}\omega_c = -\omega^{ab}$ (and similarly for J_a). However, we will find more useful to remember that $\mathrm{SO}(2,2) \simeq \mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$, which allows us to split the gauge group into two non-interacting copies, with each connection defined in the Lie algebra associated to each copy. This results in

$$I_{\text{EH}}[e,\omega] \simeq I_{\text{CS}}[A] - I_{\text{CS}}[\bar{A}] \tag{3.3}$$

(which is an equality up to boundary terms), and the connections that yield this result are

$$A_\mu = \left(\omega_\mu{}^a + \frac{1}{\ell}e_\mu{}^a\right)J_a, \quad \bar{A}_\mu = \left(\omega_\mu{}^a - \frac{1}{\ell}e_\mu{}^a\right)J_a \quad \in \mathfrak{sl}(2,\mathbb{R}) \tag{3.4}$$

3.1.1 Chern-Simons theory

Before proceeding, it is worth explaining the main features of the Chern-Simons action (3.2). This action is derived from the more general action

$$I \propto \int_{\mathcal{N}} \mathrm{Tr}(F \wedge F), \tag{3.5}$$

for a general 4d manifold $\mathcal{N}$, where $F = \mathrm{d}A + A \wedge A$ is the field strength associated to the connection A, sometimes called its curvature. It is a simple exercise to show that the integrand is a total derivative: $F \wedge F = \mathrm{d}(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A)$. Hence, if the manifold has a 3d boundary $\mathcal{M} = \partial\mathcal{N}$, then the action can be written as (3.2).

Notice an important fact: in common gauge theories (e.g. QCD, QED) the Yang-Mills Lagrangian $\mathcal{L} \propto F \wedge \star F \propto F_{\mu\nu}F^{\mu\nu}$ also contains a product of field strengths. As the field strength is gauge invariant, both the Yang-Mills and the Chern-Simons Lagrangians will be gauge invariant. However, there is an important difference between the two theories: the Yang-Mills Lagrangian requires a metric (hidden in the definition of the Hodge star, i.e. in the raising of indices), while the Chern-Simons one does not! This has the remarkable consequence of the Chern-Simons path integral (3.5) being a topological invariant and the observables computed from that action will also be topological invariants.

However, as we said the correspondence between (3.2) and (3.5) is through a total derivative, hence the action (3.2) might not be gauge invariant. If we consider a quantum gravity partition function, with this action in the exponent, the story is slightly different. The requirement that the partition function be gauge invariant implies that the Chern-Simons level k has to be quantized. In the present discussion this is not something we care about, however the equivalence between the Einstein-Hilbert and the Chern-Simons actions does impose a condition on the level k relating it to Newton's constant, namely $k = \ell/4G$.

The equations of motion corresponding to this action can be derived from both Lagrangians. In the light of (3.5), it is trivial to obtain

$$F = 0 \Rightarrow F^a = \mathrm{d}A^a + \frac{1}{2}\epsilon^a{}_{bc} A^b \wedge A^c = 0 \tag{3.6}$$

(this can also be derived from (3.2) varying the connection). Therefore, Chern-Simons theory can be thought of as a theory of flat connections, meaning that the connections satisfying the equations of motion are locally "pure gauge":

$$A = g^{-1}\mathrm{d}g. \tag{3.7}$$

The obstructions for this local statement to be global are the holonomies, as we will explain when discussing the BTZ thermodynamics in Chern-Simons language.

As announced at the beginning of the chapter, diffeomorphisms are equivalent to gauge transformations within this setting. Let us now demonstrate it. Consider the Lie derivative of the connection,

$$\mathscr{L}_\xi A = \mathrm{d}(\xi \cdot A) + \xi \mathrm{d}A = D_A(\xi \cdot A) + \xi \cdot F, \quad D_A(\lambda) = \mathrm{d}\lambda + [\lambda, A] \tag{3.8}$$

On-shell $F = 0$, hence it corresponds to a gauge transformation with parameter $\lambda^a = \xi^\mu A_\mu{}^a$.[1]

At this point two questions arise naturally: Why doesn't this work in four dimensions and why has the gauge action turned out to be the Chern-Simons one? To answer the first one, for

[1] It is more tedious, but it can also be shown for the metric. For an explicit derivation, see [13].

simplicity in four dimensional Minkowski spacetime, it should be noted that the Einstein-Hilbert action can be written as $\int e \wedge e \wedge R$, implying that an action of the form $\int A \wedge A \wedge (\mathrm{d}A + A^2)$, for some connection A, would be needed, but there is no action like this one in gauge theory, i.e. it is not gauge invariant. As for why this gauge action is the Chern-Simons one, this fact shouldn't be that surprising: as we saw in Chapter 2, three dimensional gravity doesn't have bulk degrees of freedom, therefore a theory with flat connections is the best candidate to represent it.

3.1.2 Equations of motion

From now until the end of the chapter, our task will be to reproduce the results described in the previous section, from a gauge theory perspective. The reason behind this change of viewpoint will become clear after we introduce higher spin fields coupled to gravity in the next chapter. In short, the main issue in those settings is that the metric is no longer a gauge invariant object and therefore anything built from it might even be meaningless, hence we need gauge independent tools.

The first example of this that we will show is the derivation of the equations of motion of gravity (Einstein's equations and the torsion-free condition) from a gauge independent perspective. Chern-Simons' equations of motion are, as we mentioned earlier, flat connections, therefore $F[A] = 0$ and $F[\bar{A}] = 0$. We can combine them to obtain the desired result,

$$F^a[A] + F^a[\bar{A}] = 2R^a - \Lambda \epsilon^{abc} e_b \wedge e_c = 0, \tag{3.9}$$

which are the Einstein's equations in the vacuum and

$$\frac{1}{2\ell}(F^a[A] - F^a[\bar{A}]) = \mathrm{d}e^a + \epsilon^{abc} e_b \wedge \omega_c = T^a = 0, \tag{3.10}$$

which is the torsion-free condition, both in terms of frame fields.

In the following sections we will take a slightly different route than in the last chapter: we will first study the notion of an Asymptotically Anti-de Sitter spacetime and its properties from a gauge theory perspective, to then move on to the particular case of the BTZ black hole.

3.2 Asymptotically Anti-de Sitter connections

It has already been stressed that we are mainly interested in spacetimes that asymptote to Anti-de Sitter. We must first deal with Chern-Simons theory in the presence of a boundary, in order to do this we will follow reviews such as the ones found in [9, 15]. Assuming the spacetime has the topology of a cylinder, $\mathcal{M} = \mathbb{R} \times D$, where $\mathbb{R}$ is the time direction and the boundary of D is $\partial D = S^1$, we can write the action in Hamiltonian (2+1) form,

$$I_{\mathrm{CS}} = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{d}t \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j \, \mathrm{Tr}(A_t F_{ij} - A_i \dot{A}_j) \tag{3.11}$$

From this we learn that A_i are the dynamical fields of the theory, while A_t is a Lagrange multiplier. The variation of this action is not zero, it leaves us with the following boundary contribution

$$\delta I_{\mathrm{CS}} = \int_{\mathcal{M}} (\text{e.o.m.})\, \delta A - \frac{k}{4\pi} \int_{\mathbb{R}\times S^1} \mathrm{d}x^+ \wedge \mathrm{d}x^- \, \mathrm{Tr}(A_+ \delta A_- - A_- \delta A_+), \tag{3.12}$$

where we have introduced lightcone coordinates $x^{\pm} = t \pm \phi$. The first choice we will impose, such that the boundary term vanishes and which is sensibly motivated by the gravity description, is

$$A_{-}\big|_{\text{boundary}} = 0. \tag{3.13}$$

We also need to fix the gauge: for that we impose, without loss of generality,

$$A_{\rho} = b^{-1}(\rho)\partial_{\rho} b(\rho). \tag{3.14}$$

The equations of motion $F_{\rho\phi} = F_{\rho t} = 0$ then imply

$$A_{\phi}(t,\rho,\phi) = b^{-1}(\rho) a_{\phi}(t,\phi) b(\rho), \tag{3.15a}$$

$$A_{t}(t,\rho,\phi) = b^{-1}(\rho) a_{t}(t,\phi) b(\rho), \tag{3.15b}$$

however (3.13) forces them to be $A_{\phi} = A_t$ at the boundary, and the above equations imply that this is also the case in the bulk, at least on-shell, therefore

$$A_{-} \equiv 0 \quad \text{everywhere}. \tag{3.16}$$

Asymptotic Anti-de Sitter

The most general metric for an AAdS_3 spacetime was given in the previous chapter, (2.13). One can construct the connections associated to this spacetime as

$$\begin{aligned} A &= b^{-1}(\rho)\left(L_1 + \frac{2\pi}{k}\mathcal{L}(x^+)\,L_{-1}\right) b(\rho)\,\mathrm{d}x^+ \;+\; b^{-1}(\rho)\,\partial_{\rho}\, b(\rho)\,\mathrm{d}\rho \\ \bar{A} &= -b(\rho)\left(L_{-1} + \frac{2\pi}{k}\,\bar{\mathcal{L}}(x^-)\,L_1\right) b^{-1}(\rho)\,\mathrm{d}x^- \;+\; b(\rho)\,\partial_{\rho}\, b^{-1}(\rho)\,\mathrm{d}\rho\,, \end{aligned} \tag{3.17}$$

where $b(\rho) = e^{\rho L_0}$. It is no surprise that A satisfies all the above conditions, specifically (3.13) and (3.14) in the whole space and analogously $\bar{A}$. This allows us to define the "reduced connections" $a(x^+)$, $\bar{a}(x^-)$, as those whose gauge transforms are $A, \bar{A}$, i.e.

$$\begin{aligned} A &= b^{-1} a(x^+)\, b + b^{-1}\,\mathrm{d}b \\ \bar{A} &= b\,\bar{a}(x^-)\, b^{-1} + b\,\mathrm{d}b^{-1}\,. \end{aligned} \tag{3.18}$$

From this point of view, the radial coordinate of the spacetime "emerges" as a gauge transformation of the spacetime defined by the reduced connections.

3.2.1 Asymptotic symmetries

We want to find gauge transformations leaving the conformal structure (which is defined by the connections at the boundary) invariant. This is an analogous statement to diffeomorphisms leaving the boundary metric invariant, in Chern-Simons language. As we're looking at this problem with hindsight, the answer must be that the algebra of allowed gauge transformations leaving the AdS boundary invariant must be two copies of the Virasoro algebra. Nevertheless it will be instructive to find it explicitly.

Firstly, notice that the two conditions needed to be satisfied, (3.16) and (3.14), are not enough to satisfactorily parameterize the whole space of AAdS solutions. This can be clearly seen if one acts on (3.17) repeatedly with gauge transformations, finding new connections whose shape is different to (3.17) although still satisfying the constraints. The reason for this is that diffeomorphisms are a subclass of all possible gauge transformations of the connections. In [15] a recipe is given to bypass the characterization of this subclass: they impose *ad hoc* another boundary condition for the connections, namely

$$(A - A_{\text{AdS}})\big|_{\text{boundary}} = O(1), \tag{3.19}$$

and similarly for $\bar{A}$. These conditions are natural in the sense that their requirement is analogous to imposing the Conformal Group (already existent in the boundary of pure AdS) be contained within the full Asymptotic Symmetry Group.

We can now effectively compute the algebra. Following [15], a generic AAdS (reduced) connection satisfying all above constraints can be cast in the form

$$a = L_{-1} + \frac{2\pi}{k}\mathcal{L}L_{-1}. \tag{3.20}$$

Let us apply the most general gauge transformation, with gauge parameter

$$\lambda = \sum_{i=-1}^{1} \varepsilon^i L_i\,, \tag{3.21}$$

therefore $\delta a = \mathrm{d}a + [\lambda, a]$, but restricting the parameters $\varepsilon^1\,, \varepsilon^0\,, \varepsilon^{-1}$ such that the structure of (3.20) is preserved. They then need to satisfy

$$\varepsilon^0 = -\varepsilon', \quad \varepsilon^{-1} = \frac{1}{2}\varepsilon'' + \frac{2\pi}{k}\varepsilon\mathcal{L}, \tag{3.22}$$

where we define $\varepsilon \equiv \varepsilon^1$. Under the gauge transformation with these "constrained" parameters, $\mathcal{L}$ varies as

$$\delta_\varepsilon \mathcal{L} = \varepsilon\mathcal{L}' + 2\varepsilon'\mathcal{L} + \frac{k}{4\pi}\varepsilon''', \tag{3.23}$$

which is precisely the transformation law of the stress tensor, (2.19)! From this one can compute the charges generating this transformation

$$Q(\lambda) = \int d\theta \epsilon(\theta)\mathcal{L}(\theta), \tag{3.24}$$

which we can use to infer the Poison algebra structure via the transformation

$$\delta_\lambda F = \{Q(\lambda), F\} \tag{3.25}$$

Following this procedure, and expanding $\mathcal{L}$ in Fourier modes, one finds an exact match with (2.15).

The above procedure can be followed analogously for the barred (right-sector) connection, which confirms the presence of Vir $\otimes$ Vir as the Asymptotic Symmetry Algebra.

3.3 BTZ black hole

In order to describe the BTZ black hole we will now need a pair of connections $A, \bar{A}$ such that the metric they correspond to is that of the BTZ solution. One can work them out taking the inverse procedure: one begins with a metric, reads off the vielbeins, plugs them in the Maurer-Cartan structure equations (A.14) and solves for the spin connection components, hence having all the information needed to construct the connections. The resulting connections for general M, J are (3.17), with constant $\mathcal{L}, \widetilde{\mathcal{L}}$ "modes", as expected from the metric formulation (2.14).

Thermodynamical results for the BTZ can be expressed now in terms of these "modes"

$$\beta = \frac{1}{T} = \frac{\pi}{2}\left(\sqrt{\frac{k}{2\pi\mathcal{L}}} + \sqrt{\frac{k}{2\pi\bar{\mathcal{L}}}}\right), \quad S = 2\pi\sqrt{2\pi k\mathcal{L}} + 2\pi\sqrt{2\pi k\bar{\mathcal{L}}}. \tag{3.26}$$

3.3.1 Holonomies and horizons

The question we want to answer now is the following: if we are given two connections $A, \bar{A}$, how do we know they correspond to a black hole? As discussed in subsection 2.2.2, a black hole's defining feature is the existence of an event horizon which is completely smooth when we turn to Euclidean signature. This definition of a black hole will enable us to answer the question above.

In gauge invariant language, the sensible way to describe a smooth horizon is requiring the connection to be single-valued when translated along the (Euclidean) time direction. But we can take this definition a step further, making it more precise. Firstly, in Euclidean signature the topology of the BTZ black hole is that of a solid torus, with the contractible cycle corresponding to the angular direction and the non-contractible cycle corresponding to the time direction. Secondly, the notion of translating a 1-form around a closed loop has a name: its holonomy. For instance, we can consider translating vector valued 1-forms in $T_p\mathcal{M}$ around loops with base point p, and comparing the initial and final 1-forms (we can do it as they belong to the same tangent space) which will yield an element of the isometry group of the space.

The Holonomy group is then a collection of all group elements obtained by translating the connection around any loop. In the case at hand, the connections being $SL(2,\mathbb{R})$-valued implies that their holonomies will form a subgroup of $SL(2,\mathbb{R})$. The way to compute them is through a path ordered exponential of the connection around the loop, i.e.

$$\mathrm{Hol}_\tau(A) = \mathcal{P}\exp\left\{\oint_\tau A\right\} = \mathbb{I} + \int_0^\beta A_\tau \mathrm{d}\tau + \int_0^\beta \int_0^{\tau_1} A_\tau(\tau_1)A_\tau(\tau_2)\mathrm{d}\tau_2\mathrm{d}\tau_1 + \ldots \tag{3.27}$$

where $\mathcal{P}$ is the path-ordering symbol, meaning that A being a matrix (non-commutative) renders the exponential being just a shorthand, not a real matrix exponential, as one must care about the order of integration. However, note that in the case of constant A the path ordering doesn't matter, yielding a proper matrix exponential.

Joining both ingredients (the Euclidean topology of the BTZ and the notion of holonomy), the natural requisite seems to be imposing the holonomy of the connection to be trivial, i.e. to belong to the center of the gauge group, around the non-contractible cycle of the geometry:

$$\mathrm{Hol}_\tau(A),\ \mathrm{Hol}_\tau(\bar{A}) \quad \in Z(SL(2)). \tag{3.28}$$

This statement is usually known as "the holonomy condition". For the BTZ this is actually the case: if one computes explicitly the holonomies of its connections (which are constant) around the thermal cycle, the result is

$$\mathrm{Hol}_\tau(A) = b^{-1}\mathrm{Hol}_\tau(a)b = b^{-1}e^{\omega}b = b^{-1}e^{2\pi(\tau a_z + \bar{\tau} a_{\bar{z}})}b, \quad \tau = \frac{i\beta}{2\pi}, \tag{3.29}$$

If we now plug in the value of the reduced connections and the relation between charges $\mathcal{L}, \bar{\mathcal{L}}$ and sources $\tau, \bar{\tau}$ (3.26), we find $\mathrm{Hol}_\tau(A) = \mathrm{Hol}_\tau(\bar{A}) = -\mathbb{I}$, which is indeed an element in the center of the gauge group[2].

Note that the alluded relation between charges and sources is something coming from the metric formulation. Here we want to do better: instead of assuming it, it can be viewed as a consequence of the holonomy condition, as imposing this condition will lead to the correct thermodynamic relation, in a gauge invariant manner. In the case of the BTZ this choice is arbitrary, and not very insightful as it doesn't teach us anything we didn't know before, however it will be our guiding light in the next section as in that case we will not have any thermodynamical knowledge beforehand.

[2]The fact that it is minus the identity and not the identity itself is pointed out and explained in [16], it is related to the presence of fermions in the spacetime, which we do not deal with here.

Chapter 4

Higher spin theories

In this chapter we will introduce another ingredient which is included in this project: higher spin fields coupled to gravity. We will briefly review the general theory of higher spin fields, from the motivations and issues to their basic (free field) description, and then switch to their coupling to gravity in three dimensions, where their study becomes drastically more tractable thanks to the Chern-Simons formulation. By then it should be clear why were all the concepts and jargon introduced in the last chapter so important, as the metric formulation will no longer be useful in these settings.

Although the introduction to three dimensional higher spin gravity is usually covered in any paper related to the subject, for the first part (general higher spin theory) we will be following mainly [15, 17, 18].

4.1 Higher spin theory

Higher spin particles where initially studied by Dirac, Wigner, Fierz & Pauli, etc. Since their beginning, they have been cause of physicist's headaches for several reasons. To mention some of them,

- The Coleman-Mandula no-go theorem: *The only allowed symmetries of the S-matrix are*

  $$G = G_{\text{Poincare}} \times G_{\text{gauge}}. \tag{4.1}$$

  This is the easiest problem to circumvent, as this no-go theorem is based on the quantum field theory being embedded in Minkowski space. Indeed, if we study higher spin fields in, say, AdS spacetime, this theorem no longer holds as we do not know what the S-matrix is. The same applies if one considers this theorem's supersymmetric version, the Haag–Łopuszański–Sohnius theorem.

- Interactions:

  This is the biggest issue when studying higher spin fields, as it has been proven difficult to introduce consistent interactions between them, or with other particles, while maintaining the degrees of freedom of the free theory. The way to find a consistent higher spin theory has been researched for a long time, with a successful description due to M. Vasiliev [19], and requires the introduction of an infinite tower of higher spins and higher derivative terms (in the action and the gauge transformations) to be consistently gauge invariant.

Having dealt with the issues, let's have a look at some reasons why one could be interested in them:

- Composite particles:

  Nowadays, composite particles found in collider experiments are treated with form-factors, but that is valid in the regime where exchanged momenta is bigger than the respective masses. In the cases where it is not, a description by local actions is needed.

- Limit of massless String Theory:

  Higher spin theory is closely related to String Theory: in the tensionless limit $T \sim 1/\alpha' \to 0$, one finds that the spectrum of the theory is composed of, again, an infinite tower of particles with higher spins (as $M^2 \sim T$). For this reason, it is conjectured that String Theory is nothing but a spontaneously broken phase of an underlying higher spin gauge theory, and understanding them could improve our understanding of areas such as M-theory or AdS/CFT.

- Holography:

  In recent years there has been an increasing interest in understanding higher spin theories due to their interplay with holography, as they have provided some interesting examples of such dualities. The first of these conjectures was found by I. R. Klebanov and A. M. Polyakov relating O(N) vector models with Vasiliev's higher spins in AdS_4 [20] and later on it was generalized for three dimensions by M. R. Gaberdiel and R. Gopakumar [21] as a duality between three dimensional higher spin theories and certain higher spin minimal models.

We will refer only to massless bosonic higher spin fields in the remaining.

4.1.1 Formulation of free higher spin theories

During the rest of this section we will give a short review on the free theory of higher spin fields in 4-dimensional Minkowski spacetime, although we will see that the structure of the equations does not depend on the dimension.

A successful formulation of these fields was given by C. Fronsdal [22], who found the equation of motion corresponding to the propagation of free massless spin-s fields to be the vanishing of the Fronsdal operator $\mathcal{F}_{\mu_1\ldots\mu_s}$,

$$\mathcal{F}_{\mu_1\ldots\mu_s} \equiv \Box\varphi_{\mu_1\ldots\mu_s} - \partial_{(\mu_1|}\partial^\lambda\varphi_{|\lambda\ldots\mu_s)} + \partial_{(\mu_1}\partial_{\mu_2}\varphi_{\mu_3\ldots\mu_s)\lambda}{}^\lambda = 0, \tag{4.2}$$

where $\varphi_{\mu_1\ldots\mu_s}$ is a fully symmetric rank-s tensor describing the field and the parenthesis denote complete symmetrization of the indices without normalization factors. This equation is left invariant by the following transformation

$$\delta\varphi_{\mu_1\ldots\mu_s} = \partial_{(\mu_1}\xi_{\mu_2\ldots\mu_s)}, \tag{4.3}$$

where the gauge parameter $\xi_{\mu_1\ldots\mu_{s-1}}$ must be traceless (for $s > 3$), i.e.

$$\xi_{\mu_1\ldots\mu_{s-3}\lambda}{}^\lambda = 0. \tag{4.4}$$

For $s \geq 4$ there is yet another requirement such that the degrees of freedom are the same as the free field ones, the so-called double-traceless constraint[1] $\varphi_{\mu_1 \ldots \mu_{s-4}\lambda\rho}{}^{\lambda\rho} = 0$. Not only that, but it is also the only way to obtain the equations of motion via a variational principle.

To give an idea of the reasoning that lead to the above equations, it is useful to remember the well-known lower spin ($s = 0, 1, 2$) cases:

Field	Equation of motion	Gauge redundancy
Scalar ϕ	$\partial_\mu \partial^\mu \phi = 0$	None
Maxwell field A_μ	$\partial^\mu F_{\mu\nu} = 0$	$\delta A_\mu = \partial_\mu \xi$
Graviton $g_{\mu\nu}$ (linearized)	$R_{\mu\nu} = 0$	$\delta g_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

Table 4.1: List of equations of motion and gauge transformations for the free $s = 0, 1, 2$ fields

Except for the scalar field, the rest are gauge fields with gauge redundancy eliminating spurious degrees of freedom. The Fronsdal equation (4.2) arises as the natural generalization of this table if we maintain the assumption that the equations be of second order, and the gauge redundancy (4.3) of said equation is also a generalized version of those of the lower spin cases.

All the above can be modified to introduce an AdS background (see for instance [15]), hence a coupling to gravity. In three dimensions, tensors of rank greater than one do not propagate any local degrees of freedom, as it happened with the metric. But as we know from that case, the asymptotic dynamics of such system are far from trivial. We will review these arguments from the point of view of Chern-Simons theory in the next section.

4.2 Higher spin gravity

(Yet) another miracle of three-dimensional gravity is the fact that in this setting, the infinite tower of states necessary for consistency in higher dimensions can be truncated to any finite amount (up to N) higher spin fields coupled to gravity. In the Chern-Simons formulation, this is achieved just by enhancing the gauge group to be $\mathrm{SL}(N,\mathbb{R}) \times \mathrm{SL}(N,\mathbb{R})$. This is the main motivation to study the structure of spin-3 degrees of freedom coupled to gravity, as they will provide insights for the full theory, avoiding unnecessary complications.

In order to accommodate these spin-3 fields in the frame formalism, the notions of "generalized vielbein" $e_\mu{}^{ab}$ and "generalized (dual) spin connection" $\omega_\mu{}^{ab}$ are introduced. It can be shown [15] that the linearized fluctuations of this generalized vielbein corresponds to the Fronsdal equations of motion of a higher spin field.

Their gauge indices now transform under the gauge group $\mathrm{SL}(3,\mathbb{R})$, which can be realized through generators $\{J_a, T_{ab}\}_{a,b=1,2,3}$, with T_{ab} antisymmetric. It is, though, more useful to get rid of the antisymmetric constraint by parameterizing the generators as $\{L_i, W_n\}_{i=0,\pm1,\, n=0,\pm1,\pm2}$, whose $\mathfrak{sl}(3,\mathbb{R})$ Lie algebra can be found in Appendix B. An explicit representation of this algebra (the fundamental one), which we will use quite often, can also be found there.

The action for this theory will still be given by the Chern-Simons one, with the connections living in the "enhanced" gauge group, while the metric and the higher spin fields are still defined

[1] The presence of constraints such as the ones described is an odd feature of the theory, as it seems they have to be implemented by hand. Some suggestions have been made to reconcile this using either auxiliary fields satisfying certain equations of motion, allowing for higher order equations of motion or renouncing locality.

as:

$$g_{\mu\nu} = \frac{1}{2}\text{Tr}(e_\mu e_\nu), \quad \varphi_{\mu\nu\gamma} = \frac{1}{9}\text{Tr}(e_{(\mu}e_\nu e_{\gamma)}). \tag{4.5}$$

The interpretation of the Chern-Simons theory can be made more explicit as the coupling of a spin-3 field to gravity, presenting the full non-linear action, obtained by expanding the Chern-Simons action in terms of the vielbein and spin connection

$$\begin{aligned} I \;=\; & \frac{1}{8\pi G}\int_{\mathcal{M}} \left\{ e^a \wedge \left(\mathrm{d}\omega_a + \frac{1}{2}\,\epsilon_{abc}\,\omega^b \wedge \omega^c - 2\,\epsilon_{abc}\,\omega^{bd} \wedge \omega^c{}_d \right) \right. \\ & \left. - 2\,e^{ab} \wedge \left(\mathrm{d}\omega_{ab} + \epsilon_{cd(a|}\,\omega^c \wedge \omega_{|b)}{}^d \right) + \frac{1}{6\,\ell^2}\,\epsilon_{abc} \left(e^a \wedge e^b \wedge e^c - 12\,e^a \wedge e^{bd} \wedge e^c{}_d \right) \right\}. \end{aligned} \tag{4.6}$$

and field equations; those corresponding to the gravitational fields are generalizations of the Einstein's equations and the Torsion-free condition, with the spin-3 contributions

$$\mathscr{T}^a \equiv \mathrm{d}e^a + \epsilon^{abc}\,\omega_b \wedge e_c - 4\,\epsilon^{abc}\,e_{bd} \wedge \omega_c{}^d = 0, \tag{4.7a}$$

$$\mathscr{R}^a \equiv \mathrm{d}\omega^a + \frac{1}{2}\,\epsilon^{abc}\left(\omega_b \wedge \omega_c + \frac{e_b \wedge e_c}{\ell^2}\right) - 2\,\epsilon^{abc}\left(\omega_{bd} \wedge \omega_c{}^d + \frac{e_{bd} \wedge e_c{}^d}{\ell^2}\right) = 0. \tag{4.7b}$$

Similarly we can get equations of motion for the spin-3 fields, $\mathscr{T}^{ab} = \mathscr{R}^{ab} = 0$, which we do not write explicitly but can be found in [15].

Before moving on, notice that enlarging the gauge group has an important consequence that should not be overlooked: the spacetime metric is only invariant under the $SL(2,\mathbb{R})$ subgroup: a general $SL(3,\mathbb{R})$ gauge transformation will mix the metric and the spin-3 fields in non-trivial ways. The consequence of this fact is that geometry in the presence of higher spin is not something we can rely on; quantities or notions built from the metric are no longer gauge invariant. We will discuss these ideas further when we study higher spin black holes.

4.2.1 Embeddings

A remarkable feature of $SL(N,\mathbb{R})$ gravity is the presence of different embeddings, which is the way to include the $SL(2,\mathbb{R})$ subgroup inside the full $SL(N,\mathbb{R})$ [16, 23, 25, 24]. From the physics point of view this choice of embedding will result in a different low energy field content and determine the AdS_3 vacuum around which linearized equations of motion are constructed. These embeddings are not related by conjugation, therefore defining distinct extensions of General Relativity in three dimensions. In the $SL(3,\mathbb{R})$ case, the most relevant in this thesis, there are two distinct embeddings: principal and diagonal (in $SL(N,\mathbb{R})$ those embeddings are also present, among others).

The principal embedding is the most natural to construct out of the two and it was the one used above. It corresponds to considering the generators $\{L_0, L_{\pm1}\}$ to be those of the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra. The remaining generators $\{W_0, W_{\pm1}, W_{\pm2}\}$ fall into spin-3 representations of $\mathfrak{sl}(2,\mathbb{R})$[2]. The spectrum (field content) in this embedding consists of a spin-2 field (the metric) and a spin-3 field (satisfying Fronsdal's equation at the linearized level) (4.5).

The diagonal embedding is a rather interesting higher spin extension of General Relativity, as its spectrum comprises fields of "lower spin", in the sense that it clearly contains an spin-2 field

[2] See subsection B.2.1.

(the metric), but the rest have spin lower than 2. This embedding is where the $SL(2,\mathbb{R})$ subgroup is chosen to be a block diagonal 2×2 matrix in $SL(N,\mathbb{R})$. The remaining generators transform in $2(N-2)$ spin-1/2[3] and $(N-2)^2$ spin-0 representations of $SL(2,\mathbb{R})$. This embedding has an important truncation, when the connection lies in the block diagonal $SL(2,\mathbb{R})\times U(1)^{N-1}$, therefore these solutions correspond to AdS_3 along with $2(N-1)$ flat U(1) connections.

Specifically, for $N=3$ the way to realize this embedding is by taking the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra to be that spanned by $\{L_0, L_{\pm 1}\} \to \{\frac{1}{2}L_0, \pm\frac{1}{4}W_{\pm 2}\}$. The spin-0 representation of $SL(2,\mathbb{R})$ is given by W_0 and comprises the $U(1)$, spin-1 fields of the spectrum. The spin-1/2 representations of $SL(2,\mathbb{R})$ are given by $\{L_1, W_{-1}\}$ and $\{L_{-1}, W_1\}$, and are responsible for spin-3/2 excitations of the spectrum.

We should mention that, as noted in [24], the diagonal embedding is not unitary: it contains negative norm states, as the current dual to the bulk gauge field has a Kac-Moody algebra with negative level k. For now this should not pose a problem, as throughout this and the next chapter we will only perform semiclassical analysis of the theory.

4.2.2 Asymptotic symmetries

With AdS/CFT in mind, we will be interested in having any sort of spacetime, such that its asymptotics are those of Anti-de Sitter. This was reviewed in Section 3.2, where we were careful when discussing the constraint (3.19); concretely we argued that the naturalness of this condition comes from the fact that it enforces the Asymptotic Symmetry Group to *contain* the Conformal Group $SO(2,2)$. Indeed, this constraint allows for more general groups, as is the case in higher spin gravity.

In any case, the discussion in the aforementioned section paves us the way to quickly recap the results in this more general setting, as they are obtained analogously to how it was previously explained. They lead to the most general asymptotically Anti-de Sitter connections in this new setting being

$$\begin{aligned} a(x^+) &= \left(L_1 + \frac{2\pi}{k}\mathcal{L}(x^+)L_{-1} - \frac{\pi}{2k}\mathcal{W}(x^+)W_{-2}\right)\mathrm{d}x^+, \quad A = b^{-1}a(x^+)\,b + b^{-1}\mathrm{d}b \\ \bar{a}(x^-) &= -\left(L_{-1} + \frac{2\pi}{k}\bar{\mathcal{L}}(x^-)L_1 - \frac{\pi}{2k}\overline{\mathcal{W}}(x^-)W_2\right)\mathrm{d}x^-, \quad \bar{A} = b\,\bar{a}(x^-)\,b^{-1} + b\,\mathrm{d}b^{-1} \end{aligned} \tag{4.8}$$

with the usual $b(\rho) = e^{\rho L_0}$. Notice that now there is a new set of left- and right-moving charges, subleading in the $\rho\to\infty$ limit, $\mathcal{W}, \overline{\mathcal{W}}$ and we can foresee they will have a certain transformation law, as the stress tensor charges $\mathcal{L}, \bar{\mathcal{L}}$ did. As in the $SL(2,\mathbb{R})$ case, we can apply an infinitesimal gauge transformation to this connection to find constraints between the parameters of the transformation such that we don't leave the space of AAdS connections. Under the gauge transformation with the constrained parameters (two of them independent, ε, χ), the left-moving charges trans-

[3] In three dimensions there is no Spin-Statistics theorem, which allows us to have half-integer spin bosons while still having a Hamiltonian bounded from below

form as

$$
\begin{aligned}
\delta_\varepsilon \mathcal{L} &= \varepsilon \mathcal{L}' + 2\varepsilon' \mathcal{L} + \frac{k}{4\pi}\varepsilon''' , \\
\delta_\varepsilon \mathcal{W} &= \varepsilon \mathcal{W}' + 3\varepsilon' \mathcal{W} , \\
\delta_\chi \mathcal{L} &= 2\chi \mathcal{W}' + 3\chi' \mathcal{W} , \\
\delta_\chi \mathcal{W} &= -\frac{1}{3}\left[2\chi\mathcal{L}''' + 9\chi'\mathcal{L}'' + 15\chi''\mathcal{L}' + 10\chi'''\mathcal{L} + \frac{k}{4\pi}\chi^{(5)} + \frac{64\pi}{k}(\chi\mathcal{L}\mathcal{L}' + \chi'\mathcal{L}^2)\right] ,
\end{aligned} \tag{4.9}
$$

where as expected we find in the first two equations (which are the ones defining the behaviour of the fields under conformal transformations) the transformation law of the stress tensor (2.19) and that of a primary field of conformal weight 3, respectively.

The algebra (in terms of the Fourier modes) that corresponds to these transformations is, through a straightforward generalization of (3.24) and (3.25), found to be

$$ i\left\{\mathcal{L}_p, \mathcal{L}_q\right\} = (p-q)\,\mathcal{L}_{p+q} + \frac{c}{12}(p^3 - p)\,\delta_{p+q,0} , \tag{4.10a} $$

$$ i\left\{\mathcal{L}_p, \mathcal{W}_q\right\} = (2p-q)\,\mathcal{W}_{p+q} , \tag{4.10b} $$

$$ i\left\{\mathcal{W}_p, \mathcal{W}_q\right\} = \frac{1}{3}\Big[(p-q)(2p^2+2q^2-pq-8)\,\mathcal{L}_{p+q} + \frac{96}{c}(p-q)\,\Lambda_{p+q} + \frac{c}{12}p(p^2-1)(p^2-4)\,\delta_{p+q,0}\Big] , \tag{4.10c} $$

where Λ_p is defined as

$$ \Lambda_p \equiv \sum_{q\in\mathbb{Z}} \mathcal{L}_{p+q}\mathcal{L}_{-q} . \tag{4.11} $$

This algebra goes by the name of classical[4] $\mathcal{W}_3$-algebra and it is a non-linear extension to the Virasoro algebra, containing a spin-2 generator (the stress-tensor) and a spin-3 generator (the $\mathcal{W}$ field). The same algebra is found for the right-moving modes, therefore we find the Asymptotic Symmetry Algebra of spin-3 gravity to be $\mathcal{W}_3 \otimes \mathcal{W}_3$ instead of Vir $\otimes$ Vir.

This analysis can be carried out for the diagonal embedding [26], resulting in the Bershadsky-Polyakov algebra, also known as $\mathcal{W}_3^{(2)}$-algebra, therefore finding the Asymptotic Symmetry Algebra of the spin-3 gravity in the diagonal embedding to be $\mathcal{W}_3^{(2)} \times \mathcal{W}_3^{(2)}$.

4.2.3 Black holes and the lack of geometry

It is interesting to think about black hole solutions in higher spin gravity, as they will provide useful insights on the features of the spacetime in the presence of these higher spin fields. Indeed, we have been claiming throughout this thesis the breakdown of traditional geometric notions such as horizons, curvature, etc, and higher spin black holes are *the* example of this feature. We thus need some way to describe them in a gauge invariant manner.

First of all, it should be noted that there is a trivial black hole in higher spin gravity: the usual BTZ embedded in spin-3 gravity, i.e. the connections (3.17) with constant charges $\mathcal{L}, \bar{\mathcal{L}}$ and $\mathfrak{sl}(3,\mathbb{R})$-valued. It is of course a black hole solution, a boring one nonetheless. What we are seeking here is a higher spin black hole, carrying higher spin charge along with mass and angular momentum.

[4] The quantum version involves modifications coming from the normal ordering of the modes in Λ_p. We do not worry about that here.

Before studying actual examples of black hole solutions in higher spin gravity it is worth understanding how to define them, in other words, what do we expect from them before even attempting to find them. In the literature a set of guiding principles have been put forward with this purpose [16, 27, 28, 29]

1. They should have a smooth BTZ limit. This is a natural condition as the BTZ is also a solution of the theory, with mass and angular momentum independent of the higher spin charges.

2. They should have an event horizon in Lorentzian signature, smooth in Euclidean. As explained in the previous chapter, we will translate these conditions to the Chern-Simons formulation through the use of holonomies.

3. They should have a consistent thermodynamical interpretation. In this context we will expect the solutions to satisfy certain integrability conditions.

We will discuss these requirements in more detail as we go along. We should also mention that in the principal embedding we will focus on the first higher spin black hole discovered [27], although it is now known that there are other solutions such as the one found in [23], which exhibit different properties.

Principal embedding

Let us discuss a black hole solution actually carrying spin-3 charges. Its systematic characterization was carried out in [27, 28] and reviewed in [29]. However, soon after this discovery, a tension between the results computed originally and those computed for the same solution by other methods appeared. Over the next years this dissonance has been well understood and resolved [25, 30].

In the original paper [27], the first higher spin black hole solution was constructed by invoking the AdS/CFT correspondence: a $\mathcal{W}_3 \times \mathcal{W}_3$-extended CFT has by definition a dimension (3,0) primary $\mathcal{W}$ and a dimension (0,3) primary $\overline{\mathcal{W}}$. They are included in the action as

$$I_{\mathrm{CFT}} \to I_{\mathrm{CFT}} + \int \mathrm{d}x^2 \mu(x)\mathcal{W}(x) + \int \mathrm{d}x^2 \bar{\mu}(x)\overline{\mathcal{W}}(x), \tag{4.12}$$

therefore they are a deformation of the CFT. However, notice that this is an irrelevant deformation, as the operators have scaling dimension 3. Via the AdS/CFT dictionary, sources are identified with boundary conditions in the bulk spacetime. This means that the notion of an Asymptotically AdS_3 spacetime analyzed in the previous subsection is not applicable: it needs to be generalized. They propose reduced connections of the form

$$\begin{aligned} a &= \left(L_1 - \frac{2\pi}{k}\mathcal{L}L_{-1} - \frac{\pi}{2k}\mathcal{W}W_{-2}\right)\mathrm{d}x^+ + \left(\mu W_2 + w_1 W_1 + w_0 W_0 + w_{-1}W_{-1} + w_{-2}W_{-2} + \ell L_{-1}\right)\mathrm{d}x^-, \\ \bar{a} &= -\left(L_{-1} - \frac{2\pi}{k}\bar{\mathcal{L}}L_1 - \frac{\pi}{2k}\overline{\mathcal{W}}W_2\right)\mathrm{d}x^- - \left(\bar{\mu}W_{-2} + \bar{w}_{-1}W_{-1} + \bar{w}_0 W_0 + \bar{w}_1 W_1 + \bar{w}_2 W_2 + \bar{\ell}L_1\right)\mathrm{d}x^+ \end{aligned} \tag{4.13}$$

and whose full connection is $A = b^{-1}ab + b^{-1}\mathrm{d}b$ with the same radial gauge function $b(\rho)$ (likewise for the barred connection). The meaning of these new generalized AAdS_3 connections is that they correspond to a different UV (due to the introduction of the irrelevant operators). While the undeformed theory asymptotes to AdS_3 with radius ℓ, the deformed one corresponds to a theory with $\ell/2$.

All the unspecified functions $\mathcal{L}, \mathcal{W}, \mu$, etc can depend on $x^{\pm}$, but not ρ. They are also not independent, they are constrained by the bulk equations of motion (the flatness of the connections). Through the use of Ward identities the authors of [27] are able to show that $\mathcal{L}, \mathcal{W}$ correspond to the deformed CFT's spin-2 and spin-3 primaries, respectively, while μ corresponds to the spin-3 source. This was also necessary for a consistent holographic dictionary of such solutions.

A black hole solution is required to be stationary, therefore all the parameters in the solution are constant. Once this is taken into account and after imposing the equations of motion, one finds the black hole solution[5]

$$\begin{aligned} a &= \Big(L_1 - \frac{2\pi}{k}\mathcal{L}L_{-1} - \frac{\pi}{2k}\mathcal{W}W_{-2}\Big)\mathrm{d}x^+ + \mu\Big(W_2 - \frac{4\pi\mathcal{L}}{k}W_0 + \frac{4\pi^2\mathcal{L}^2}{k^2}W_{-2} + \frac{4\pi\mathcal{W}}{k}L_{-1}\Big)\mathrm{d}x^-, \\ \bar{a} &= -\Big(L_{-1} - \frac{2\pi}{k}\bar{\mathcal{L}}L_1 - \frac{\pi}{2k}\overline{\mathcal{W}}W_2\Big)\mathrm{d}x^- + \bar{\mu}\Big(W_{-2} - \frac{4\pi\bar{\mathcal{L}}}{k}W_0 + \frac{4\pi^2\bar{\mathcal{L}}^2}{k^2}W_2 + \frac{4\pi\overline{\mathcal{W}}}{k}L_1\Big)\mathrm{d}x^+. \end{aligned} \tag{4.14}$$

Clearly, these connections satisfy requirement 1, as they have a smooth BTZ limit, namely that with all higher spin parameters turned off: $\mu, \bar{\mu}, \mathcal{W}, \overline{\mathcal{W}} \to 0$. It is straightforward to compute the metric associated to these connections[6], which indeed goes as $e^{4\rho/\ell}$ for $\mu\bar{\mu} \neq 0$ instead of $e^{2\rho/\ell}$, as observed before.

- Integrability condition: higher spin black holes are to be thought of as contributing to a generalized partition function, including the chemical potentials conjugate to $\mathcal{L}, \mathcal{W}$ and the barred counterparts.

$$Z[\tau, \alpha, \bar{\tau}, \bar{\alpha}] = \mathrm{Tr}\, e^{4\pi^2 i[\tau\mathcal{L} - \bar{\tau}\bar{\mathcal{L}} + \alpha\mathcal{W} - \bar{\alpha}\overline{\mathcal{W}}]} = \mathrm{Tr}_{\mathrm{CFT}}\, q^{L_0 - \frac{c}{24}}\bar{q}^{\bar{L}_0 - \frac{c}{24}}u^{W_0}\bar{u}^{\bar{W}_0} \tag{4.15}$$

  where the last equality follows from the AdS dictionary, with $q \equiv e^{2\pi i\tau}$, $u \equiv e^{2\pi i\alpha}$. Here τ and $\alpha = \bar{\tau}\mu$ are potentials conjugate to the $\mathcal{L}, \mathcal{W}$ charges, respectively. In the BTZ case, the solution is labelled by either τ or $\mathcal{L}$, as they are related. Here we need a similar assignment $(\mathcal{L}, \mathcal{W}) \to (\tau\alpha)$. The charges, from the CFT point of view, are expectation values

$$\mathcal{L} = \langle\mathcal{L}\rangle = -\frac{i}{4\pi^2}\frac{\partial \ln Z}{\partial\tau}, \quad \mathcal{W} = \langle\mathcal{W}\rangle = -\frac{i}{4\pi^2}\frac{\partial \ln Z}{\partial\alpha}. \tag{4.16}$$

  This leads to a constraint, needed for a consistent thermodynamical interpretation

$$\frac{\partial\mathcal{L}}{\partial\alpha} = \frac{\partial\mathcal{W}}{\partial\tau}. \tag{4.17}$$

  The fulfillment of this constraint is equivalent to saying that the black hole obeys the first law of thermodynamics, therefore satisfying requirement 3. Needless to say, this discussion applies analogously to the barred quantities.

- Absence of conical singularity vs holonomy condition: let us discuss the non-rotating solution, hence $\bar{\mu} = -\mu$, $\bar{\mathcal{L}} = \mathcal{L}$, $\overline{\mathcal{W}} = -\mathcal{W}$. From the connections (4.14) one can construct the

[5]These connections are sometimes seen in the literature written in Euclidean signature, which amounts to sending $x^+ \to z$, $x^- \to -\bar{z}$.

[6]We will not display it here as it does not give us any more insights than (4.18), the non-rotating version.

metric

$$
\begin{aligned}
&\mathrm{d}s^2 = \mathrm{d}\rho^2 - \mathcal{F}(\rho)\mathrm{d}t^2 + \mathcal{G}(\rho)\mathrm{d}\phi^2,\\
&\mathcal{F}(\rho) = \left(2\mu e^{2\rho} + \frac{\pi}{k}\mathcal{W}e^{-2\rho} - \frac{8\pi^2}{k^2}\mu\mathcal{L}^2 e^{-2\rho}\right)^2 + \left(e^{\rho} - \frac{2\pi}{k}\mathcal{L}e^{-\rho} + \frac{4\pi}{k}\mu\mathcal{W}e^{-\rho}\right)^2,\\
&\mathcal{G}(\rho) = \left(e^{\rho} + \frac{2\pi}{k}\mathcal{L}e^{-\rho} + \frac{4\pi}{k}\mu\mathcal{W}e^{-\rho}\right)^2 + 4\left(\mu e^{2\rho} + \frac{\pi}{2k}\mathcal{W}e^{-2\rho} + \frac{4\pi^2}{k^2}\mu\mathcal{L}^2 e^{-2\rho}\right)^2 + \frac{4}{3}\left(\frac{4\pi}{k}\right)^2\mu^2\mathcal{L}^2,
\end{aligned}
\tag{4.18}
$$

and find where the $g_{tt} = \mathcal{F}(\rho)$ component vanishes; that is, where the event horizon is located. There are two possibilities, the first one corresponds to the BTZ limit ($\mathcal{W} = 0$), which we dismiss as it is not what we are looking for. The other one requires

$$
k + 32\mu^2\pi(\mu\mathcal{W} - \mathcal{L}) = 0, \tag{4.19}
$$

so that the g_{tt} component indeed vanishes. As in the BTZ case, one can expand the metric around the event horizon, which is located at $e^{\rho_+} = \sqrt{(2\pi\mathcal{L} - 4\pi\mu\mathcal{W})/k}$, and read off the temperature after demanding the absence of a conical singularity. That along with (4.19) leads to problematic values of $\mathcal{W}, \mathcal{L}$ in terms of μ, β: on the one hand, they diverge when $\mu \to 0$ (which contradicts requirement 1), on the other hand they violate the integrability condition, therefore contradicting requirement 3.

It is clear that the absence of a conical singularity, a procedure working without obstructions for the BTZ, is not valid in this context anymore. The holonomy condition prescribed in the previous chapter is brought into play now, shining due to its gauge invariance. Consider the holonomy of the connections (back to the rotating case) around the identified Euclidean coordinates $(z, \bar{z}) \sim (z + 2\pi\tau, \bar{z} + 2\pi\bar{\tau})$. As the reduced connections are constant they exponentiate dismissing the path ordering, while the radial function β appears as a conjugation

$$
\mathrm{Hol}_{\tau,\bar{\tau}}(A) = b^{-1}e^{\omega}b, \quad \omega = 2\pi(\tau a_+ - \bar{\tau}a_-) \tag{4.20a}
$$

$$
\mathrm{Hol}_{\tau,\bar{\tau}}(\bar{A}) = be^{\bar{\omega}}b^{-1}, \quad \bar{\omega} = 2\pi(\tau\bar{a}_+ - \bar{\tau}\bar{a}_-) \tag{4.20b}
$$

Indeed, in [27] it is claimed that the correct characterization of a smooth horizon is represented by the imposition that the holonomy is trivial in the sense that the eigenvalues of $\omega, \bar{\omega}$ are demanded to be $(2\pi i, 0, -2\pi i)$ as in the BTZ. To remove the ambiguity of the ordering of these eigenvalues, it is better to package this information in the following equations:

$$
\det(\omega) = 0, \quad \mathrm{Tr}(\omega^2) + 8\pi^2 = 0, \quad \det(\bar{\omega}) = 0, \quad \mathrm{Tr}(\bar{\omega}^2) + 8\pi^2 = 0. \tag{4.21}
$$

These conditions, for the connections (4.14) lead to

$$
0 = 2048\pi^2\alpha^3\mathcal{L}^3 - 576\pi k\tau^2\alpha\mathcal{L}^2 - 864\pi k\alpha^2\tau\mathcal{W}\mathcal{L} - 864\pi k\alpha^3\mathcal{W}^2 - 27k^2\tau^3\mathcal{W}, \tag{4.22a}
$$

$$
0 = -256\pi^2\alpha^2\mathcal{L}^2 - 24\pi k\tau^2\mathcal{L} - 72\pi k\tau\alpha\mathcal{W} - 3k^2 \tag{4.22b}
$$

and similar for the barred counterparts. By suitably applying derivatives to these equations with respect to α, τ, one can show that the charges $\mathcal{L}, \mathcal{W}$ do satisfy the integrability condition (4.17).

The authors use this consistency check as strong evidence[7] that the holonomy condition is the correct characterization of the black hole charges.

However this has an important implication: this charge assignment leads to a metric (4.18) which does not have an event horizon! This solution is actually named the "wormhole solution", and we can actually gauge transform it to one carrying an event horizon [28]. This, however, renders the notion of an event horizon meaningless. There is therefore no analogue of the area law in higher spin gravity, but that does not mean that we cannot compute the black hole's entropy.

For what follows it will be interesting to solve (4.22), leaving the solution in terms of a parameter C as

$$\mathcal{W} = \frac{4(C-1)}{C^{3/2}}\mathcal{L}\sqrt{\frac{\mathcal{L}}{k}}, \quad \mu = \frac{3\sqrt{C}}{4(2C-3)}\sqrt{\frac{k}{\mathcal{L}}}, \quad \mathcal{L} = \frac{k\pi^2}{\beta^2}\frac{C(3-2C)^2}{(C-3)^2(4C-3)}. \tag{4.23}$$

The BTZ limit (requirement 1) is achieved in the $C \to \infty$ limit.

- Black hole entropy

There are different ways to compute the entropy of a higher spin black hole; when these computations were carried out two distinct results for said entropy were found, depending on the method used. We will now briefly show why it arises and later explain how it is understood. In [25] this quantity is computed via the free energy $F = -T\log Z$, where the partition function is obtained from the Euclidean on-shell action (supplemented by suitable boundary terms). They found that this entropy takes the form

$$S = 2\pi k \mathrm{Tr}_f[(\lambda_\phi - \bar{\lambda}_\phi)L_0], \tag{4.24}$$

where λ_ϕ is the eigenvalue matrix of a_ϕ, likewise for $\bar{\lambda}_\phi$. The entropy just depends on the eigenvalues along the non-contractible circle ϕ, which is natural from the point of view of the holonomy characterizing the black hole.

If one evaluates it for the non-rotating case ($\bar{\lambda}_\phi = -\lambda_\phi$), with eigenvalues

$$\lambda_\phi = 2\sqrt{\frac{\mathcal{L}}{kC}}\mathrm{diag}\left(\frac{3+C(\sqrt{4C-3}-2)}{2C-3}, 2, \frac{3-C(\sqrt{4C-3}-2)}{2C-3}\right), \tag{4.25}$$

the result found is

$$S_{\text{can}} = 4\pi\sqrt{\mathcal{L}}\left(1-\frac{3}{2C}\right)^{-1}\sqrt{1-\frac{3}{4C}}, \tag{4.26}$$

which does have a good BTZ limit, $C \to \infty$. This is the so-called canonical entropy. However this disagrees with the entropy obtained in [27] from the partition function (4.15), which is

$$S_{\text{holo}} = 4\pi\sqrt{\mathcal{L}}\sqrt{1-\frac{3}{4C}}. \tag{4.27}$$

[7] They also check that the triviality of the holonomy leads to a correct smoothness condition for the linearized spin-3 field, providing further support for the claim. See [27] for the details.

This entropy is instead captured by [25]

$$S = 2\pi k \mathrm{Tr}[(\lambda_z - \bar{\lambda}_{\bar{z}})L_0] \tag{4.28}$$

and is referred to as holomorphic entropy.

This difference can be traced back to the μ-deformation required for the black hole thermodynamical description, which we discussed before. In the original paper [27] the deformation is introduced in the Lagrangian (4.12). However the deformation can be introduced in the Hamiltonian instead

$$H_{\mathrm{CFT}} \to H_{\mathrm{CFT}} + \int \mathrm{d}\phi \mu \mathcal{W} + \int \mathrm{d}\phi \bar{\mu} \overline{\mathcal{W}}. \tag{4.29}$$

The former is more natural from the CFT point of view due to the holomorphic and anti-holomorphic structures of 2d CFT, the latter is more natural in the bulk, if one considers constant time slices where the Hamiltonian is defined. Both answers are backed up with different computations, which depend on whether they rely on the Lagrangian or the Hamiltonian. The reason for the discrepancy is because the relation between the Lagrangian and Hamiltonian is via a Legendre transform, which is non-trivial if one considers spin-3 or higher deformations.

Diagonal embedding

It is also interesting to discuss the diagonal embedding black hole due to its simplicity and striking similarity to the BTZ. Such a solution was first found in [16] and has the following connections

$$\begin{aligned} a &= (W_2 + \omega\, W_{-2} - q\, W_0)\, \mathrm{d}x^+ + \frac{\eta}{2}\, W_0\, \mathrm{d}x^-, \quad b(\rho) = e^{\rho L_0}, \\ \bar{a} &= (W_{-2} + \omega\, W_2 - q\, W_0)\, \mathrm{d}x^- + \frac{\eta}{2}\, W_0\, \mathrm{d}x^+. \end{aligned} \tag{4.30}$$

This solution corresponds to a black hole with mass w, whose metric is

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 - 4\left(e^{2\rho} - we^{-2\rho}\right)^2 \mathrm{d}t^2 + 4\left(e^{2\rho} + we^{-2\rho}\right)^2 \mathrm{d}\phi^2\,, \tag{4.31}$$

charged under the U(1) fields

$$\chi = -q\mathrm{d}x^+ + \frac{\eta}{2}\mathrm{d}x^-\,, \quad \bar{\chi} = -q\mathrm{d}x^- + \frac{\eta}{2}\mathrm{d}x^+\,. \tag{4.32}$$

The trivial holonomy condition yields

$$\exp\left(\int_0^{2\pi\beta} A_\tau \mathrm{d}\tau\right)\,, \quad \exp\left(\int_0^{2\pi\beta} \bar{A}_\tau \mathrm{d}\tau\right) \;\to\; \eta = 2q\,, \quad \beta = \frac{1}{8\sqrt{w}}\,. \tag{4.33}$$

The thermodynamical analysis for this black hole can be carried out, finding the usual Bekenstein-Hawking entropy

$$S = \frac{2\pi\ell r_+}{4G} = \frac{8\pi\ell\sqrt{w}}{4G} = \frac{A}{4G}\,, \tag{4.34}$$

In this case there is no issue like that of the principal embedding, as the field content in this embedding does not include spin-3 charges, therefore the Legendre transform between the Hamiltonian and Lagrangian is trivial.

Chapter 5

Wilson lines in Einstein gravity

In 2013, two alternative proposals to the Ryu-Takayanagi formula [1] (relating the Entanglement Entropy of a CFT to the area of a minimal surface in AdS whose endpoints lie on the CFT boundary) were raised for the case of three-dimensional gravity, one of them by Jan de Boer and Juan I. Jottar [3], and the other one by Martin Ammon, Alejandra Castro, and Nabil Iqbal [2]. Here we will focus on the latter (although they were later proven to be equivalent [32]). The importance of these proposals lie in the explicit use of the topological features of three dimensional gravity, instead of the geometrical ones, which might be inaccessible in other settings, as we will explore in the next chapter.

We will begin by introducing Wilson lines, from basic definitions to their suitability when it comes to describe geodesics in three dimensional gravity. Using a prescription given in [2], we will provide evidence of the connection between the saddle point (semiclassical) approximation of these Wilson lines and the entanglement entropy, ascertaining the equivalence between the proposal and the Ryu-Takayanagi formula. We will then turn to understand the quantum mechanical features of the Wilson line away from the semiclassical approximation. As the goal of this thesis is performing the quantum mechanical study for higher spin gravity, in this chapter we will skip several computations and discussions which, although being interesting, will not be relevant later on, and can be found in the pertaining papers.

Some knowledge on the basics of entanglement entropy will be required at certain points, as well as the Ryu-Takayanagi proposal. For a short review we refer the reader to Appendix C or [43] and the references therein.

5.1 Wilson lines as topological probes

Wilson lines are a main ingredient in gauge theories such as QCD [31], as well as natural observables in Chern-Simons theory. They are defined as

$$W_{\mathcal{R}}(C) = \mathrm{Tr}_{\mathcal{R}}\left[\mathcal{P}\exp\left(-\int_C \mathcal{A}\right)\right], \quad \mathcal{A} \in \mathfrak{g}, \tag{5.1}$$

where $\mathcal{R}$ is the representation of the gauge group G, whose corresponding algebra is $\mathfrak{g}$, and C is the open interval within $\mathcal{M}$ along which the connection is translated. If the curve C is closed, then it is called a Wilson loop, and it is simple to show using Stokes' theorem that it is a gauge-invariant quantity. The topological nature of three dimensional gravity will also entail that the path itself doesn't play a role, only topological data such as open/closed, endpoints or windings.

The choice of representation will specify the role played by the Wilson line as a geometric probe, as we will see it plays the role of a geodesic in the bulk spacetime. We will seek a massive probe, meaning that we need an infinite dimensional representation (besides the fact that there are no unitary finite-dimensional representations of $\mathfrak{sl}(2,\mathbb{R})$). We will make use of the highest-weight representation of $\mathfrak{sl}(2,\mathbb{R})$, defined from a highest weight state satisfying

$$\ell_0 |\text{hw}\rangle = h |\text{hw}\rangle , \quad \ell_1 |\text{hw}\rangle = 0, \quad |h,n\rangle \sim (\ell_{-1})^n |\text{hw}\rangle . \tag{5.2}$$

This infinite tower of states forms an unitary, irreducible and infinite-dimensional representation, labelled by its quadratic Casimir. The Casimir element is an invariant of the representation, therefore the same for any state within it. It is then computed as

$$c_2 = \langle \text{hw}|C_2|\text{hw}\rangle = \langle \text{hw}|\eta^{ab}\ell_a\ell_b|\text{hw}\rangle = 2h(h-1), \tag{5.3}$$

where η_{ab} is the Killing form of $\mathfrak{sl}(2,\mathbb{R})$, given in (B.4). From the AdS/CFT perspective this corresponds to a massive particle in the bulk dual to a CFT operator of dimensions $(h,\bar{h})$, with mass $m = h+\bar{h}$, and spin $s = h-\bar{h}$.

5.1.1 Generalized geodesics

These infinite-dimensional representations are common in quantum mechanics. For convenience, we will make use of an auxiliary quantum mechanical system for some field U with a global symmetry G, living on the Wilson line such that upon quantization the Hilbert space of the system is the representation we look for. This was shown by Witten [34], and concretely reads[1]

$$W_{\mathcal{R}}(C) = \int \mathcal{D}U \exp[-S(U;\mathcal{A})_C], \quad S(U;\mathcal{A})_C = S(U)_{C,\text{free}} + S(U;\mathcal{A})_{C,\text{int}} \tag{5.4}$$

(where $\mathcal{A}$ is the $\mathfrak{so}(2,2)$-valued connection). The (first order) action for a massive particle moving in an $\text{SL}(2,\mathbb{R})$ manifold is given by [35]

$$S(U,P)_{\text{free}} = \int_C \mathrm{d}s \left[\text{Tr}\left(PU^{-1}\frac{\mathrm{d}U}{\mathrm{d}s}\right) + \lambda(s)\left(\text{Tr}P^2 - c_2\right)\right], \tag{5.5}$$

where $U \in \text{SL}(2,\mathbb{R})$, $P \in \mathfrak{sl}(2,\mathbb{R})$ is the momentum conjugate to U, s is an affine parameter, Tr denotes contraction with the Killing form and $\lambda(s)$ is a Lagrange multiplier constraining the norm of the momentum. This action has the appropriate global symmetry $\text{SL}(2,\mathbb{R})\times\text{SL}(2,\mathbb{R})$:

$$U(s) \to LU(s)R, \quad P(s) \to R^{-1}P(s)R, \quad L,R \in \text{SL}(2,\mathbb{R}). \tag{5.6}$$

The coupling to gravity is made through the usual minimal coupling prescription, promoting (5.6) to a local gauge symmetry along the worldline, embedded in the space defined by the connections $A \in \mathfrak{sl}(2,\mathbb{R})$, $\bar{A} \in \overline{\mathfrak{sl}(2,\mathbb{R})}$, which are invariant under

$$A \to L(x)(A+\mathrm{d})L^{-1}(x), \quad \bar{A} \to R^{-1}(x)(\bar{A}+\mathrm{d})R(x), \tag{5.7}$$

[1]Note that from here on, for a consistent notation with the papers involved, we denote the action by S instead of I.

hence introducing the covariant derivative

$$\frac{\mathrm{d}U}{\mathrm{d}s} \mapsto D_s U = \frac{\mathrm{d}U}{\mathrm{d}s} + A_s U - U\bar{A}_s, \quad A_s \equiv A_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s}, \tag{5.8}$$

ensures the local gauge symmetry for the action. This action, written in the second order formulation (integrating out the conjugate momentum P in (5.5)), has the form

$$S(U; A, \bar{A}) = \sqrt{c_2} \int_C \mathrm{d}s \sqrt{\mathrm{Tr}(U^{-1} D_s U)^2}. \tag{5.9}$$

One can vary this action with respect to the field U, finding

$$\frac{\mathrm{d}}{\mathrm{d}s}\left((A^u - \bar{A})_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right) + [\bar{A}_\mu, A^u_\nu]\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0, \quad A^u \equiv U^{-1}\left(A_s + \frac{\mathrm{d}}{\mathrm{d}s}\right)U \tag{5.10}$$

If one fixes $U = \mathbb{I}$ in this equation, the geodesic equation emerges when $A, \bar{A}$ are expressed in terms of the vielbein and spin connection. In some sense the choice of U is indicating the probe which path to follow within the bulk; the trivial choice $U = \mathbb{I}$ yields the geodesic path. But this choice of path is irrelevant (up to topologically inequivalent paths) due to the flatness of the connections, hence this result is general. Not only that, by inserting $U = \mathbb{I}$ in (5.9) we get

$$S(U; A, \bar{A}) = \sqrt{c_2} \int_C \mathrm{d}s \sqrt{\mathrm{Tr}(A_s - \bar{A}_s)^2} = \sqrt{2c_2} \int_C \mathrm{d}s \sqrt{g_{\mu\nu}(x)\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}}, \tag{5.11}$$

which is the proper length along the geodesic. This implies that in the saddle point approximation of the Wilson line

$$W_{\mathcal{R}}(C) \approx \exp(-S_{\text{on-shell}}) = \exp\left(-\sqrt{2c_2} L_C\right), \tag{5.12}$$

where L_C is the geodesic distance. This observation is the starting point of the analogy the authors make with the Ryu-Takayanagi formula and is backed up with a computation of the backreaction of the Wilson line on the geometry which we do not carry out here but refer to Section III.A of [2]; what we will require from it is the effective assignment

$$\sqrt{2c_2} \to \frac{c}{6}, \tag{5.13}$$

relating the strength of the probe (c_2) and the central charge of the boundary CFT (c). We will now turn to show that this prescription correctly computes entanglement entropies of the CFT.

5.2 Semiclassical computations

It is necessary to check whether the prescription given above is correct or not. Proving it in all generality is an almost impossible challenge, instead in [2] they choose to compute the entanglement entropies of settings where we already know they answer, to see if they match. These checks are non trivial, and in order to do them they develop some techniques that exploit the $\mathrm{SL}(2,\mathbb{R})$ invariance of the probe[2], which we will briefly review as will be of relevance later on.

[2] As it is mentioned in the paper, this is not the only route available for this computation. However, the fact that it exploits the topological nature of the system rather that its geometrical features is an advantage, as by now the reader should foresee that the aim is to generalize it in due time to higher spins.

5.2.1 Open lines attached to the boundary

To begin, consider the (first order) probe action,

$$S(U,P;A,\bar{A}) = \int_C \mathrm{d}s \left[\mathrm{Tr}\left(PU^{-1}D_sU\right) + \lambda(s)\left(\mathrm{Tr}P^2 - c_2\right)\right], \quad U \in \mathrm{SL}(2,\mathbb{R}),\ P \in \mathfrak{sl}(2,\mathbb{R}) \tag{5.14}$$

with equations of motion

$$U^{-1}D_sU + 2\lambda P = 0, \quad \frac{\mathrm{d}P}{\mathrm{d}s} + [\bar{A}_s, P] = 0, \quad \mathrm{Tr}P^2 = c_2. \tag{5.15}$$

where $\mathrm{Tr}P^2 \equiv \delta_{ab}P^aP^b$. On-shell, this action reduces to

$$S_{\mathrm{on-shell}} = -2c_2 \int_C \mathrm{d}s\lambda(s). \tag{5.16}$$

The name of the game now consists of finding the simplest way to compute the value of $\lambda(s)$, on-shell.

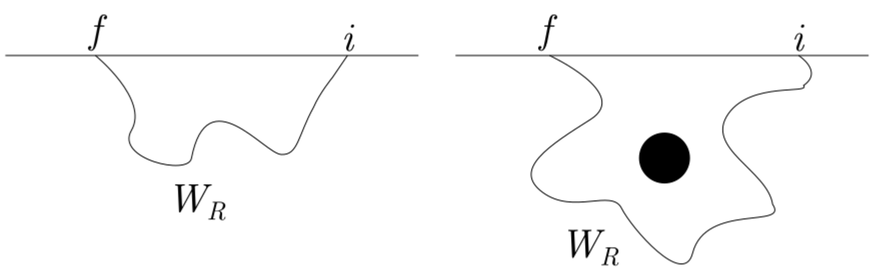

Figure 5.1: Cartoon of the boundary-to-boundary Wilson lines studied here, the left case corresponds to pure AdS while the right one corresponds to the presence of the BTZ. Both describe a fixed time slice.

We will compute λ for two spacetimes whose boundary CFT is well known, as well as their respective entanglement entropies: pure AdS_3 (in Poincaré coordinates), dual to a vacuum state in the CFT defined on the plane, and the BTZ black hole, dual to a thermal state in the CFT. Let us now introduce some notation here: $s \in (0, s_f)$ will be the parameter along the trajectory and the boundary conditions for the Wilson line are

$$\rho(s_f) = \rho(0) \equiv \rho_0, \quad \phi(s_f) - \phi(0) \equiv \Delta\phi, \tag{5.17}$$

and we need to define an UV cutoff scale $\epsilon \equiv e^{-\rho_0}$, as $\rho_0 \to \infty$ at the boundary. Also, keep in mind that the entanglement entropy is computed on a fixed time slice, represented in Figure 5.1. This will imply that in the BTZ case there is no winding dependence in the Wilson line, if there were the line would have to cross itself.

The way to solve for λ consists of the following: as we're provided with flat connections, they are locally a gauge transform of the "nothingness", $A = \bar{A} = 0$. Therefore we can solve (5.15) in this trivial gauge, which yields

$$U_0(s) = u_0 \exp(-2\alpha(s)P_0), \quad \frac{\mathrm{d}\alpha}{\mathrm{d}s} = \lambda, \quad \mathrm{Tr}P_0^2 = c_2, \tag{5.18}$$

with u_0, P_0 constant elements, labelling the solutions. Note that now the interest has shifted to computing α, as $S_{\text{on-shell}} = -2c_2\Delta\alpha$. We now need to gauge transform the connections to match those of the actual metric:

$$\begin{aligned} &A = L\mathrm{d}L^{-1}, \quad L(x) = b(\rho)^{-1}\exp\left(-\int_{x_0}^{x} a\right), \quad b(\rho) = e^{\rho L_0}, \\ &\bar{A} = R^{-1}\mathrm{d}R, \quad R(x) = \exp\left(\int_{x_0}^{x} \bar{a}\right) b(\rho)^{-1}, \end{aligned} \tag{5.19}$$

where $a, \bar{a}$ are the reduced connections. This gauge transform changes the solution too: now the full fledged solution is given by (5.6)

$$U(s) = L(x(s))\, U_0(s)\, R(x(s)), \tag{5.20}$$

(P_0 is also gauge transformed but it will not be relevant for us). We can solve for α by considering this equation at the endpoints $s = 0, s = s_f$. After some algebra,

$$\exp(-2\Delta\alpha P_0) = (R(0)U_i^{-1}L(0))\,(R(s_f)U_f^{-1}L(s_f))^{-1}. \tag{5.21}$$

Using the argued boundary conditions[3] $U_i = U_f = \mathbb{I}$, one is ready to compute alpha by noticing that the equation is representation independent. Thus, we can trace it in the fundamental representation of $\mathrm{SL}(2,\mathbb{R})$, given in Appendix B (also remember that the trace of P_0 and its square in the fundamental representation are fixed and yield eigenvalues $\pm\sqrt{c_2/2}$).

- Pure AdS_3 (Poincaré): the above procedure offers us the following solution

$$\Delta\alpha = -\frac{1}{\sqrt{2c_2}}\cosh^{-1}\left(1 + \frac{e^{2\rho_0}(\Delta\phi)^2}{2}\right). \tag{5.22}$$

  The on-shell action, upon introducing the UV cutoff $\epsilon \equiv e^{-\rho_0}$, turns out to be

$$S_{\text{on-shell}} \sim 2\sqrt{2c_2}\log(\Delta\phi/\epsilon) \tag{5.23}$$

  and the entanglement entropy computed, after the assignment (5.13), is then

$$S_{\text{EE}} = \frac{c}{3}\log\left(\frac{\Delta\phi}{\epsilon}\right). \tag{5.24}$$

  This is a well-known result from vacuum CFT_2 at zero temperature [1, 33], and a good consistency check for a generalization of Ryu and Takayanagi's proposal.

[3] This choice ensures that the probe, although sourcing curvature in the ambient space, does not source torsion, and furthermore it ensures invariance under local Lorentz rotations at the boundary

- BTZ black hole: the same procedure yields for the non-rotating solution

$$S_{\rm EE} = \frac{c}{3}\log\left[\frac{\beta}{\pi\epsilon}\sinh\left(\frac{\pi\Delta\phi}{\beta}\right)\right], \quad \beta = \pi\sqrt{\frac{k}{2\pi\mathcal{L}}}, \tag{5.25}$$

  which is the generalization of (5.23) for a system at finite temperature (which makes sense as black holes are dual to thermal states). Again, this result is what we expected from CFT [1, 33]. The rotating black hole can also be studied and the outcome corresponds to a thermal state with different left- and right-moving temperatures, as it should.

Thermal loops

We can also study Wilson loops winding around the BTZ (a topological obstruction is needed to be enclosed in the loop, otherwise it will be contractible, yielding no results). The interpretation of this computation is clear: we are computing the proper length of a Wilson loop around the horizon of the black hole. As such we expected to find Bekenstein-Hawking's entropy for the BTZ black hole.

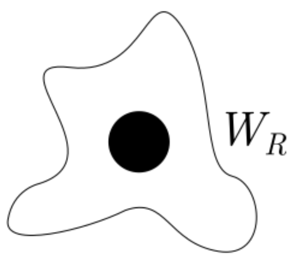

Figure 5.2: Cartoon of the Wilson loop being computed in this case. Remember that the proper length will be the same no matter the shape of the loop.

The procedure for this computation is similar in essence to that of the open Wilson line case. The difference has its root in the boundary conditions for the probe, in this case we only need to impose smoothness in the path of the probe, i.e.

$$x^\mu(s_f) = x^\mu(0), \quad U(s_f) = U(0), \quad P(s_f) = P(0). \tag{5.26}$$

The rest of the computation is analogous once we take this into account. The result for the entropy is

$$S_{\rm thermal} = 2\pi\sqrt{2c_2}\,{\rm Tr}_f[(\lambda_\phi - \bar{\lambda}_\phi)L_0], \tag{5.27}$$

where λ_ϕ is the eigenvalue matrix of a_ϕ and analogously for $\bar{\lambda}_\phi$. Restricting this to the BTZ the expression for the entropy is.

$$S_{\rm thermal} = 2\pi\sqrt{2\pi k\mathcal{L}} + 2\pi\sqrt{2\pi k\bar{\mathcal{L}}}. \tag{5.28}$$

This is precisely the Bekenstein-Hawking entropy of the BTZ, (2.11), as we already showed in (3.26).

5.3 Quantum mechanical description

Even though the semiclassical approximation used in the previous section successfully reproduced the already known results for entanglement and thermal entropies, it was exactly that: an approximation. Not only that, but it was a prescription only applicable if one wanted to compute Wilson loops or boundary-to-boundary Wilson lines. However one might be interested in other observables, such as bulk-to-bulk or bulk-to-boundary Wilson lines, therefore the picture was far from complete.

In 2018, a follow-up paper was released [4] where the authors addressed the fully-fledged quantum mechanical treatment of the $\mathrm{SL}(2,\mathbb{R})$ probe particle. Not only that, but they also understood the boundary states needed in the open Wilson line, and they showed it to be an AdS propagator. This section will be a summary of those results. It should be emphasized that the quantum description is that of the probe; we consider it within a fixed (classical) background geometry.

5.3.1 Hilbert space, boundary states and inner product

In the previous chapter, we considered a field $U(x)$ such that its dynamics upon quantization correspond to the representation $\mathcal{R}$ of the Wilson line, which was argued to be a highest weight representation. We will here consider a Hilbert space related to this field $U(x)$ whose quantum mechanical states belong to this highest weight representation. This representation was already defined in (5.2).[4]

Now, the states within the representation belong not to one, but to two copies of $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{sl}(2,\mathbb{R})\times\overline{\mathfrak{sl}(2,\mathbb{R})}$, with $h=\bar{h}$ (as we are interested in the representation of a massive scalar), hence they are denoted as $|h,k\rangle\otimes|h,\bar{k}\rangle\equiv|h,k,\bar{k}\rangle$.

With this in mind, we need to find a state which transforms as the $U(x)$ field did, (5.6). Specifically we want

$$G(L)\overline{G}(R^{-1})\,|U\rangle=|LUR\rangle\,, \tag{5.29}$$

where $G(M)$ denotes the group action. In [4], it is shown that a state satisfying

$$Q_a(U)\,|U\rangle\equiv[l_a+D_a{}^{a'}(U)\,\bar{l}_{a'}]\,|U\rangle=0, \tag{5.30}$$

also satisfies (5.29). Here $D_a{}^{a'}(U)$ are the matrices for the adjoint representation of $\mathrm{SL}(2,\mathbb{R})$, i.e. $D_a{}^{a'}(M)\,l_a=G(M^{-1})\,l_a\,G(M)$ (and analogously for the right sector).

Two examples of said states are given: they are the usual "Ishibashi" and "crosscap Ishibashi" (which we will abbreviate as just crosscap) states from boundary conformal field theory [36]:

$$[l_a-\bar{l}_{-a}]\,|\Sigma_{\mathrm{Ish}}\rangle=0\Longrightarrow|\Sigma_{\mathrm{Ish}}\rangle=\sum_{k=0}^{\infty}|h,k,k\rangle\,, \tag{5.31a}$$

$$[l_a-(-1)^a\bar{l}_{-a}]\,|\Sigma_{\mathrm{cross}}\rangle=0\Longrightarrow|\Sigma_{\mathrm{cross}}\rangle=\sum_{k=0}^{\infty}(-1)^k\,|h,k,k\rangle\,. \tag{5.31b}$$

There are some comments that arise naturally at this point:

[4] Here and in the remaining of this section we will use the convention that L_a are the general algebra generators while l_a are specifically associated to the highest weight representation, in analogy with [4].

- The group elements having the appropriate adjoint actions on $\bar{l}_a$ are

$$\Sigma_{\text{Ish}} = \exp\left\{-i\frac{\pi}{2}(L_1 - L_{-1})\right\}, \quad \Sigma_{\text{cross}} = \exp\left\{\frac{\pi}{2}(L_1 + L_{-1})\right\}. \tag{5.32}$$

- We can generate a whole family of states, dubbed "rotated Ishibashi states" by acting on these reference states with suitable $G(L)\overline{G}(R^{-1})$.

- Is there any difference between choosing any of these two states? It turns out that the crosscap state is the appropriate choice, we will elaborate on this at the end of this section.

Recall now our original goal: computing inner products of the form

$$W_{\mathcal{R}} = \langle U|G(\mathcal{P}e^{\int A})\overline{G}(\mathcal{P}e^{\int \bar{A}})|U\rangle \equiv \langle U|G(L)\overline{G}(R^{-1})|U\rangle\,. \tag{5.33}$$

Using the transformation property of the $|U\rangle$ states (5.29), we can rewrite the inner product as the overlap of a left-sector element $M = L\Sigma R\Sigma^{-1} = V\exp(-\alpha L_0)V^{-1}$. We assume that $L\Sigma R\Sigma^{-1}$ falls in this conjugacy class, which is an assumption satisfied by the bulk connections of the spacetimes considered. For both Ishibashi and crosscap, the computation yields

$$\langle\Sigma|G(L)\overline{G}(R^{-1})|\Sigma\rangle = \langle\Sigma|G(M)|\Sigma\rangle = \sum_{k=0}^{\infty} e^{-\alpha(h+k)} = \frac{e^{-\alpha h}}{1-e^{-\alpha}}. \tag{5.34}$$

It is then argued that the result is a $\mathfrak{sl}(2,\mathbb{R})$ character, depending only on the conjugacy class of M, which is controlled by α. As a side note, the result is independent of how we compute the inner product; if instead of using the transformation property of the $|U\rangle$ states to move the group elements to the left sector we had used it to move them to the right sector, there would be no change.

This result can also be extended to any rotated states continuously connected to Σ_{Ish}, Σ_{cross} as

$$\langle U_1|U_2\rangle = \frac{e^{-\alpha h}}{1-e^{-\alpha}}, \quad U_1^{-1}U_2 \equiv V\exp(-\alpha L_0)V^{-1}. \tag{5.35}$$

where it is clearly seen that α can be interpreted as a notion of distance within the $SL(2,\mathbb{R})$ group manifold, between U_1 and U_2.

5.3.2 Properties of the inner product and computations

We now turn to discuss the properties of the result (5.35) that will improve our understanding of it along with its ties to geometry and the path integral approach.

Green's function on the group manifold

As $SL(2,\mathbb{R})$ is a Lie group, it corresponds to a group manifold with coordinates σ^α, therefore has associated vector fields ${\xi_a}^\alpha$, $\bar{\xi}_a^\alpha$ generating the left and right group actions, i.e.

$${\xi_a}^\alpha\frac{\partial U(\sigma)}{\partial\sigma^\alpha} = L_a U(\sigma), \quad {\bar{\xi}_a}{}^\alpha\frac{\partial U(\sigma)}{\partial\sigma^\alpha} = U(\sigma)L_a. \tag{5.36}$$

This also applies to the $|U\rangle$ states: they were constructed such that they have the same transformation as U, therefore

$$\xi_a{}^\alpha \frac{\partial}{\partial \sigma^\alpha} |U(\sigma)\rangle = |L_a U(\sigma)\rangle = \ell_a |U(\sigma)\rangle \,. \tag{5.37}$$

Acting twice with this transformation on the inner product (5.35),

$$\eta^{ab} \langle U(\sigma_1)| \, \xi_a{}^\alpha \partial_\alpha \left(\xi_b{}^\beta \, |U(\sigma_2)\rangle \right) = \eta^{ab} \langle U(\sigma_1) \,|\, \ell_a \ell_b \,|\, U(\sigma_2)\rangle = c_2 \langle U(\sigma_1) \,|\, U(\sigma_2)\rangle \,, \tag{5.38}$$

where $U(\sigma_1) \neq U(\sigma_2)$ and the value of the quadratic Casimir of the higher weight representation $c_2 = \ell^a \ell^b \eta_{ab} = 2h(h-1)$. In group theory, the so-called invariant Laplacian $\Box$ is very closely related to the quadratic Casimir

$$\Box \equiv g^{ab} \partial_a \partial_b = g^{ab} \xi_a{}^\alpha \xi_b{}^\beta \frac{\partial}{\partial \sigma^\alpha} \frac{\partial}{\partial \sigma^\beta}, \quad g^{ab} = 2\eta^{ab}, \tag{5.39}$$

in fact it only differs in a factor of 2. Therefore

$$\left(\frac{1}{2} \Box_{U_2} - 2h(h-1) \right) \langle U_1 \,|\, U_2 \rangle = \frac{1}{8\pi} \delta(U_1, U_2), \tag{5.40}$$

where the U_2 subscript of the Laplacian means that the differentiation is with respect to the σ_2^α coordinates and $\delta(U_1, U_2)$ is a normalized[5] delta function on the group manifold, signaling the divergence appearing in (5.35) when $U_1 = U_2$.

Green's function on AdS_3: the Wilson line as a propagator

So far the discussion has dealt with algebraic properties of the Wilson line, now it is time to shift to its geometric properties, specially its ties to AdS_3. For that we rewrite the flat connections as

$$A(x) = g_L(x) \mathrm{d} g_L(x)^{-1}, \quad \bar{A}(x) = g_R(x)^{-1} \mathrm{d} g_R(x), \tag{5.41}$$

where the group elements g_L, g_R implement the path ordered exponentials in (5.33) as

$$\mathcal{P} \exp\left(-\int_\gamma A \right) = g_L(x_f) g_L(x_i)^{-1}, \quad \mathcal{P} \exp\left(-\int_\gamma \bar{A} \right) = g_R(x_f)^{-1} g_R(x_i). \tag{5.42}$$

These group elements, for constant reduced connections $a, \bar{a}$ can be explicitly written as

$$g_L(x) = b(\rho)^{-1} e^{-a_\mu y^\mu}, \quad g_R(x) = e^{\bar{a}_\mu y^\mu} b(\rho)^{-1}, \quad y^\mu = (t, \phi). \tag{5.43}$$

where $b(\rho)$ parameterizes the radial gauge. Defining the state

$$|U(x)\rangle \equiv G(g_L(x)^{-1}) \overline{G}(g_R(x)) \, |\Sigma\rangle \,, \tag{5.44}$$

with $|\Sigma\rangle$ being a suitable boundary state, such as the Ishibashi or crosscap discussed previously. The Wilson line can then be computed as

$$W_\mathcal{R} = \langle U | G(\mathcal{P} e^{\int A}) \overline{G}(\mathcal{P} e^{\int \bar{A}}) | U \rangle = \langle U(x_f) | U(x_i) \rangle \,. \tag{5.45}$$

[5] The normalization is with respect to the so-called de Haar measure on $SL(2,\mathbb{R})$, we do not discuss that and refer the reader to [4].

This implicitly assumed unitary group elements[6], $g_L^{-1} = g_L^\dagger$, $g_R^{-1} = g_R^\dagger$, and the splitting of the path at a midpoint between x_i and x_f where both $g_L = g_R = \mathbb{I}$. The computation should be clear by now: the Wilson line can be cast as (5.35) with

$$\mathscr{G}(x_i, x_f) = g_L(x_f) g_L(x_i)^{-1} \tilde{g}_R(x_i)^{-1} \tilde{g}_R(x_f) = V \exp\left(-\alpha(x_i, x_f) L_0\right) V^{-1}, \tag{5.46}$$

where

$$\tilde{g}_R \equiv \Sigma^{-1}\, g_R\, \Sigma, \quad \tilde{\bar{A}} \equiv \Sigma^{-1}\, \bar{A}\, \Sigma. \tag{5.47}$$

The Σ element has as "role" tying both left and right sectors. This interpretation is also present in boundary CFT, where these states are also present.

Equation (5.46) is representation independent, therefore the simplest way to solve it is resorting to the fundamental representation of $\mathfrak{sl}(2, \mathbb{R})$. Tracing it over this representation yields

$$\cosh\left(\frac{\alpha(x_i, x_f)}{2}\right) = \frac{1}{2} \mathrm{Tr}_f[\mathscr{G}(x_i, x_f)]. \tag{5.48}$$

which upon inserting the unitary group elements associated to AdS_3 and the BTZ, leads to the identification $\alpha(x_i, x_f) = 2D(x_i, x_f)$, where $D(x_i, x_f) = \mathrm{arcosh}(\sigma(x_i, x_f))$ is the geodesic length of the effective metric build from $A, \tilde{\bar{A}}$,

$$\begin{aligned}\sigma(x_i, x_f) = {} & 2\cosh(\Delta\rho)\cosh\left(\sqrt{C}(\Delta t - \Delta\phi)\right)\cosh\left(\sqrt{C}(\Delta t + \Delta\phi)\right) \\ & - \left(C\left(e^{\rho_f + \rho_i}\right) + \frac{e^{-\rho_f - \rho_i}}{C}\right)\sinh\left(\sqrt{C}(\Delta t - \Delta\phi)\right)\sinh\left(\sqrt{C}(\Delta t + \Delta\phi)\right).\end{aligned} \tag{5.49}$$

(where this result is related to the notation used in (3.17) through constant $C = -2\pi\mathcal{L}/k = -2\pi\bar{\mathcal{L}}/k$)

The Wilson line is then

$$W_{\mathcal{R}}(x_i, x_f) = \langle U(x_f) | U(x_i) \rangle = \frac{e^{-2hD(x_i, x_f)}}{1 - e^{-2D(x_i, x_f)}}, \tag{5.50}$$

which is the bulk-to-bulk propagator of a minimally coupled scalar field in a locally AdS_3 background and indeed agrees in the large h limit with the semiclassical result from [2].

The authors note, though, that this success relies on a specific evaluation of the Wilson line for the spacetimes mentioned. They improve this by showing that the Wilson line is also a Green's function on the metric defined by $A, \tilde{\bar{A}}$. The way to achieve this by relating the movement in such space with that on the $SL(2, \mathbb{R})$ manifold, where we already know that it is a Green's function.

Consider the group element $\mathscr{G}(x_i, x_f)$ defined above (5.46). Taking a derivative with respect to x_f yields

$$\frac{\partial}{\partial x_f^\mu} \mathscr{G}(x_i, x_f) = -A_\mu(x_f) \mathscr{G}(x_i, x_f) + \mathscr{G}(x_i, x_f) \tilde{\bar{A}}_\mu(x_f). \tag{5.51}$$

We would like to find a set of vector fields ${\zeta_a}^\mu, {\bar{\zeta}_a}^\mu$ defined on the AdS_3 geometry with the same group action (5.36),

$${\zeta_a}^\mu \frac{\partial}{\partial x_f^\mu} \mathscr{G}(x_i, x_f) = L_a \mathscr{G}(x_i, x_f), \quad {\bar{\zeta}_a}^\mu \frac{\partial}{\partial x_f^\mu} \mathscr{G}(x_i, x_f) = \mathscr{G}(x_i, x_f) L_a. \tag{5.52}$$

[6] It is easy to check that the constant connections corresponding to an Asymptotically AdS_3 spacetime given in (3.17), when exponentiated (5.43), do not satisfy this property. For the extent of [4] covered in this thesis it will not be an issue, however for certain computations later in that paper one needs to choose a different parameterization of the connections via $SL(2, \mathbb{R})$ automorphisms.

We can multiply both sides of these equations by L_b, substitute (5.51) and then trace them in the fundamental representation. The resulting equations have solutions for $\zeta_a{}^\mu$ if the "generalized vielbeins" constructed as $A - \mathscr{G}\tilde{\bar{A}}\mathscr{G}^{-1}$ and $-\mathscr{G}A\mathscr{G}^{-1} + \tilde{\bar{A}}$ are invertible. If this is satisfied the existence of these vector fields is proven along with their correspondence with the movement in the SL(2, $\mathbb{R}$) manifold. The Wilson line is then a Green's function on the metric defined by the connections,

$$\left(\frac{1}{2}\Box_{x_f} - 2h(h-1)\right) W_{\mathcal{R}} = \frac{1}{8\pi}\frac{\delta(x_i, x_f)}{\sqrt{-g}}. \tag{5.53}$$

The above is enough for a simply connected spacetime, where the answer only depends on the endpoints (x_i, x_f). If the bulk spacetime in consideration is not simply connected, there will be a non-trivial dependence on the winding number, as is the case for the BTZ black hole (provided there is a non-trivial holonomy around the horizon). To obtain a result only dependent on the endpoints, we must sum over all inequivalent paths, i.e.

$$\begin{aligned} W_{\mathcal{R}}(x_i, x_f) &= \sum_{\gamma(x_i,x_f)} W_{\mathcal{R}}(\gamma(x_i, x_f)) \\ &= \sum_{n\in\mathbb{Z}} \frac{e^{-2hD_n(x_i,x_f)}}{1 - e^{-2D_n(x_i,x_f)}}. \end{aligned} \tag{5.54}$$

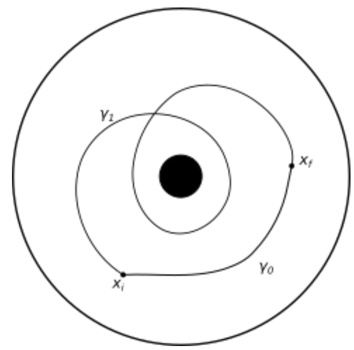

Figure 5.3: Different topologically inequivalent paths for the Wilson line in the presence of a BTZ black hole, depending on the winding: γ_0 has none ($n = 0$) while γ_1 winds once ($n = 1$). In this case there is no restriction for the path to lie on a time slice, which is why the crossing of γ_1 is allowed.

where $D_n(x_i, x_f)$ is the geodesic distance for a path around the BTZ with winding number is $n \in \mathbb{Z}$, given by

$$D_n(x_i, x_f) = \frac{1}{r_+^2}\left(r_f r_i \cosh(r_+\Delta\phi + 2\pi r_+ n) - \sqrt{(r_f^2 - r_+^2)(r_i^2 - r_+^2)}\cosh(r_+\Delta t)\right) \tag{5.55}$$

The story explained in this chapter, using Wilson lines in three dimensional gravity in order to compute either entropies or propagators, opened another window to explore these subjects, previously only studied in the metric formulation. However, this novel characterization has an important consequence: it allows for generalizations based on the Chern-Simons approach, such as higher spin theory, which will be explained in the following chapter.

Chapter 6

Wilson lines in higher spin gravity

One of the most important reasons as to why the authors pursued a generalization of the Ryu-Takayanagi proposal, relying solely on the topological features of the spacetime, is the opportunity of studying entanglement entropies in extended CFTs. In fact the closest constructions to that of [33] (making use of n-sheeted Riemann surfaces, twist fields and so on) to derive the entanglement entropy of a $\mathcal{W}_n$-extended 2d CFT are present in [38, 39], besides that not much is known about the entanglement entropy in these settings.

6.1 Wilson line as a topological probe

In the case of higher spin gravity, the Wilson line is computed as a trace of the holonomy over the highest weight representation of the $\mathrm{SL}(N,\mathbb{R})\times\mathrm{SL}(N,\mathbb{R})$ group [2, 32]. The highest weight representation is built from a highest weight state $|\mathrm{hw}\rangle$ satisfying

$$L_0\,|\mathrm{hw}\rangle = h\,|\mathrm{hw}\rangle\,,\quad W_0^{(s)}\,|\mathrm{hw}\rangle = w_s\,|\mathrm{hw}\rangle\,,\quad L_i\,|\mathrm{hw}\rangle = W_i^{(s)}\,|\mathrm{hw}\rangle = 0,\quad i>0,\quad 2<s\leq N \tag{6.1}$$

where h, w_s are the quantum numbers defining the highest weight representation. For the purpose of this chapter, i.e. computing entanglement entropy, we are interested in probes defined by a highest weight representation of $\mathfrak{sl}(N,\mathbb{R})\times\mathfrak{sl}(N,\mathbb{R})$, carrying mass but no other numbers, i.e.

$$h=\bar{h},\quad w_s=\bar{w}_s=0\quad\forall s. \tag{6.2}$$

We will postpone a detailed discussion of this representation to the next section, where it will be thoroughly studied. Instead we will now show how the construction devised in the previous chapter with the aim of replicating the Ryu-Takayanagi formula generalizes to the higher spin case.

As in the $\mathrm{SL}(2,\mathbb{R})$ case, this Wilson line can be rewritten as a path integral for the probe particle, whose action is a generalization of (5.14) including the higher order Casimirs

$$S(U,P;A,\bar{A}) = \int_C \mathrm{d}s\left[\mathrm{Tr}\left(PU^{-1}D_sU\right)+\lambda_2(s)\left(\mathrm{Tr}P^2-c_2\right)+\ldots+\lambda_N(s)\left(\mathrm{Tr}P^N-c_N\right)\right], \tag{6.3}$$

This action has a local $\mathrm{SL}(N,\mathbb{R})\times\mathrm{SL}(N,\mathbb{R})$ symmetry and presents $(N-1)$ Lagrange multipliers $\lambda_2,\ldots\lambda_N$. Tr indicates the contraction with a suitable Killing form (δ_{ab}, h_{abc}, ...), which can be found for $N = 2, 3$ in Appendix B. The equations of motion for this action are

$$U^{-1} D_s U + 2\lambda_2 P + 3\lambda_3 P \times P + ... + N\lambda_N \underbrace{P \times ... \times P}_{N-1} = 0, \quad \frac{\mathrm{d}P}{\mathrm{d}s} + [\bar{A}_s, P] = 0. \tag{6.4}$$

with the cross products of P as a shorthand for

$$\underbrace{P \times ... \times P}_{n} \equiv h_{a_1 ... a_n} P^{a_1} ... P^{a_{n-1}} T^{a_n}, \quad 2 \leq n \leq N-1, \tag{6.5}$$

where $h_{a_1 ... a_n}$ are the Killing forms of the $\mathfrak{sl}(N, \mathbb{R})$ algebra. The momentum P is also constrained due to the Lagrange multipliers (and their variation), enforcing it to satisfy

$$\mathrm{Tr} P^m \equiv h_{a_1 ... a_m} P^{a_1} ... P^{a_m} = c_m, \quad 2 \leq m \leq N. \tag{6.6}$$

The claim raised for this higher spin probe is that for an appropriate highest weight representation it computes the entanglement entropy of a certain CFT at the conformal boundary of AdS via (5.12) [2]. As in the SL(2, $\mathbb{R}$) case, at the end of the computations we will use an effective assignment relating the Casimirs and the central charge of the boundary theory, which depends on the embedding. This, again, is done by studying the backreaction of the Wilson line on the geometry, imposing it to only source curvature, but not higher spin fields nor torsion. This requires the probe to carry the quantum numbers (6.2). In the SL(3, $\mathbb{R}$) case, by computing this backreaction the following assignments are found

$$c_{2,P} \to \frac{1}{2} \left(\frac{c_P}{12} \right)^2, \quad c_{2,D} \to 2 \left(\frac{c_D}{12} \right)^2, \quad c_{3,P,D} \to 0, \tag{6.7}$$

where P denotes the principal embedding and D denotes the diagonal embedding. One can use the semiclassical (large h, w) Casimirs to find that these relations can be written in terms of the highest weights as

$$h_{P,D} \to \frac{c_{P,D}}{12}, \quad w = 0. \tag{6.8}$$

A computation of the Casimirs away from the semiclassical limit will be carried out in the next chapter. The assignment (6.8) is a better choice to specify the charges of the probe, as $c_3 = 0$ is not uniquely solved by $w = 0$, hence it will be the one used.

6.1.1 Lack of geodesic interpretation

One would like to know if we can recover the geodesic equation, or a modification of it, in the SL(N, $\mathbb{R}$) case, as we showed in subsection 5.1.1 for pure gravity. As in [2] we will gloss over some nuances, considering this to be purely illustrative.

The reasoning is the following: consider the equations of motion (6.4), choose a gauge where the Lagrange multipliers $\lambda_i(s)$, $i > 2$ are constant (the existence of such choice is one of the alluded nuances) and take a covariant derivative of the equations. We find

$$\frac{\mathrm{d}}{\mathrm{d}s} \left((A^u - \bar{A})_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \right) + [\bar{A}_\mu, A^u_\nu] \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0. \tag{6.9}$$

In the $SL(2,\mathbb{R})$ case there was an analog set of equations, which was invertible, in the sense that it was a system of three differential equations (the dimension of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$) for three components x^μ of the path. This means that we can generally find a solution. However, in the higher spin gravity case, the system is overconstrained, as the dimension of the Lie algebras is higher than the number of components of the path. We will generally not be able to find a solution, hence (5.10) can not be regarded as the geodesic equation anymore. This does not affect the rest of the analysis, it just blurs its interpretation.

6.2 Semiclassical computations

As we said earlier, this probe is supposed to compute entanglement entropies. To support this claim with evidence, we will need to compute the on-shell action,

$$S_{\text{on-shell}} = -\int \mathrm{d}s\, (2c_2\lambda_2(s) + 3c_3\lambda_3(s) + \ldots + Nc_N\lambda_N(s)) \equiv -2c_2\Delta\alpha_2 - 3c_3\Delta\alpha_3 - \ldots - Nc_N\Delta\alpha_N. \quad (6.10)$$

The goal now is computing $\Delta\alpha_m$ for given $A, \bar{A}$ connections. As in the $SL(2,\mathbb{R})$ case, we will do so by first solving the equations of motion in the empty gauge $A = \bar{A} = 0$, finding U_0 and P_0 (subject to the constraints $\mathrm{Tr}P_0^n = c_n$, $1 < n \leq N$), and later gauge transform the solution to that of the spacetime of interest. Gauge transforming this solution with L, R and imposing $U_i = U_f = \mathbb{I}$ in analogy with the $SL(2,\mathbb{R})$ case, one is left with

$$M = e^{\mathbb{P}}, \quad M \equiv R(x_i)L(x_i)L(x_f)^{-1}R(x_f)^{-1}, \quad (6.11)$$

where $\mathbb{P}$ is defined as

$$\mathbb{P} = -2\Delta\alpha_2 P_0 - 3\Delta\alpha_3 P_0 \times P_0 - \ldots - N\Delta\alpha_N P_0 \times \ldots \times P_0. \quad (6.12)$$

It can be easily noticed that we can now compute the on-shell action as

$$S_{\text{on-shell}} = \mathrm{Tr}(\mathbb{P}P_0) = \mathrm{Tr}(\log(M)P_0), \quad (6.13)$$

due to the trace constraints on P_0. For the purpose of computing entanglement entropies we need a suitable P_0 to determine the representation $\mathcal{R}$ of the Wilson line, i.e. that with mass and $w = 0$ (for the Wilson line to only source curvature and no higher spin fields). Without loss of generality it can be chosen to be

$$P_0 = \sqrt{\frac{c_2}{\mathrm{Tr}_f(L_0L_0)}}L_0. \quad (6.14)$$

This construction will be useful in both the open line and thermal loop cases, with the main difference being the modification of the conditions for the probe: as in the $SL(2,\mathbb{R})$ case we impose the conditions $U(x_f) = U(x_i)$, $P(x_i) = P(x_f)$, for them to be single valued around the loop.

6.2.1 Thermal loops

As in [2], we are going to restrict the current discussion to the spin-3 case, where we expect to find the thermal entropy of such higher spin black holes through this formalism. Following an analogous procedure as in the $SL(2,\mathbb{R})$ case, it yields

$$S_{\text{thermal}} = 2\pi\sqrt{\frac{c_2}{2}}\mathrm{Tr}_f[(\lambda_\phi - \bar{\lambda}_\phi)L_0], \quad (6.15)$$

where again λ_ϕ are the eigenvalues of a_ϕ and likewise for the barred variable. But notice that what we just found is the same as (4.24), therefore the results obtained through this formalism are those already discussed in subsection 4.2.3.

6.2.2 Open lines attached to the boundary

For the case of boundary to boundary Wilson lines, the corresponding entanglement entropy should be that of a CFT with $\mathcal{W}_3$-algebra (in the principal embedding). The strategy followed in [2] to simplify the calculations is to expand M in powers of the cutoff $\epsilon \equiv e^{-\rho}$, which enables them to find a general expression for its eigenvalues by finding the roots of the characteristic polynomial[1] of said matrix,

$$\lambda_M = \mathrm{diag}\left(\frac{m_1}{\epsilon^4}, \frac{m_2}{m_1}, \frac{\epsilon^4}{m_2}\right), \tag{6.17}$$

where m_1 and m_2 are defined in terms of the traces of M, M^2 as

$$\mathrm{Tr}_f(M) = \frac{m_1}{\epsilon^4} + O(\epsilon^{-2}), \quad \mathrm{Tr}_f(M)^2 - \mathrm{Tr}_f(M^2) = \frac{2m_2}{\epsilon^4} + O(\epsilon^{-2}). \tag{6.18}$$

For suitable $P_0 = \sqrt{c_2/2}\, L_0$, the entanglement entropy of a general spacetime is then found to be

$$S_{\mathrm{EE}} = \sqrt{2c_2}\log\left(\frac{\sqrt{m_1 m_2}}{\epsilon^4}\right). \tag{6.19}$$

We now show the results achieved with this formula

- Diagonal embedding: The vacuum solution (corresponding to the diagonal embedding black hole with $\omega = q = \eta = 0$) yields

$$S_{\mathrm{EE}} = \frac{c_D}{3}\log\left(\frac{\Delta\phi}{\epsilon^2}\right), \tag{6.20}$$

  whereas the black hole solution yields

$$S_{\mathrm{EE}} = \frac{c_D}{3}\log\left(\frac{1}{\sqrt{\omega}\epsilon^2}\sinh\left(4\Delta\phi\sqrt{\omega}\right)\right). \tag{6.21}$$

  Both solutions are expected, as they are those of the $SL(2,\mathbb{R})$ vacuum and BTZ. This is the case because of the consistent truncation of the diagonal embedding into $SL(2,\mathbb{R})$ with $U(1)$ holonomies (not present in the result, though), as with the thermal loops.

- Principal embedding: In this case the vacuum solution is clearly that of the $SL(2,\mathbb{R})$ case (5.24) with $c = c_P$. We will not display here the black hole case as it is not very insightful and in the next chapter we will find it with more generality. For the interested reader this result and discussion on its infrared and UV limits can be found in [2, 39].

[1] The characteristic polynomial of a 3×3 matrix M can be cast in the following form

$$P(\lambda_M) = -\lambda_M^3 + \mathrm{Tr}(M)\lambda_M^2 - \frac{1}{2}(\mathrm{Tr}(M)^2 - \mathrm{Tr}(M^2))\lambda_M + \det(M). \tag{6.16}$$

In our case, $\det(M) = 1$ as $M \in SL(3,\mathbb{R})$.

6.3 Quantum mechanical description

The goal of this section is to generalize what was done in section 5.3 (the quantum mechanical description of the Wilson line) for the higher spin cases, more concretely we will focus on the spin-3 case, which will provide us with enough insights (in relation to, for instance, the breakdown of geometry) and challenges in comparison with the pure gravity case, although we will attempt to address even higher ($s \geq 4$) cases along the way.

6.3.1 Hilbert space

The auxiliary quantum mechanical system living in the Wilson line will be given by an infinite dimensional representation of SL(3, $\mathbb{R}$), whose algebra is given by (B.5), again because of unitarity reasons but also because we now need two tunable parameters (mass and higher spin charge) to describe our probe. We will choose to work with a highest weight representation, and its defining highest weight state will satisfy (6.22) for $N = 3$:

$$L_0 |\text{hw}\rangle = h |\text{hw}\rangle , \quad W_0 |\text{hw}\rangle = w |\text{hw}\rangle , \quad L_1 |\text{hw}\rangle = W_1 |\text{hw}\rangle = W_2 |\text{hw}\rangle = 0, \quad . \tag{6.22}$$

There are some issues we need to address concerning this highest weight representation, which were not present in the $\mathfrak{sl}(2, \mathbb{R})$ case. Once we've dealt with them properly, we can compute the Ishibashi/crosscap states.

In order to understand the representation, we need to find the states that compose it. They are the descendants of the highest weight state, and naively they will be built by acting on it with L_n, W_n, $n < 0$, however one can check that these generators are not a diagonal basis for the Cartans L_0, W_0. Instead, we find that the suitable basis is given by

$$L_{\pm 1}, W_{\pm 1}, W_{\pm 2} \longrightarrow X_{\pm 1} = L_{\pm 1} + W_{\pm 1}, \quad Y_{\pm 1} = L_{\pm 1} - W_{\pm 1}, \quad W_{\pm 2} \tag{6.23}$$

and the explicit commutation relations with the Cartans are now

$$\begin{aligned} &[L_0, X_{\pm 1}] = \mp X_{\pm 1}, \quad [L_0, Y_{\pm 1}] = \mp Y_{\pm 1}, \quad [L_0, W_{\pm 2}] = \mp 2\, W_{\pm 2}, \\ &[W_0, X_{\pm 1}] = \mp 2\, X_{\pm 1}, \quad [W_0, Y_{\pm 1}] = \pm 2\, Y_{\pm 1}, \quad [W_0, W_{\pm 2}] = 0. \end{aligned} \tag{6.24}$$

Note that X_1, Y_1, W_2 will still annihilate the highest weight state.

Action of the raising/lowering operators

In order to solve the Ishibashi/crosscap constraints, we need to know how the coefficients of the generators acting on states are related. We will show it for the X generators, but is straightforwardly generalized for the rest.

We will label the descendants as $|h, w; k, s\rangle$, where $|h, w; 0, 0\rangle \equiv |h, w\rangle$ denotes the highest weight state and k, s are the descendant labels associated to h, w respectively. Starting with the highest weight state,

$$X_{-1} |h, w; 0, 0\rangle = \alpha_{h,w} |h, w; 1, 2\rangle . \tag{6.25}$$

If we compute the inner product of these states, knowing that X_1 annihilates the highest weight state, we find[2]

$$
\begin{aligned}
|\alpha_{h,w}|^2 \langle h,w;1,2|h,w;1,2\rangle &= \langle h,w;0,0|X_{-1}^\dagger X_{-1}|h,w;0,0\rangle = \langle h,w;0,0|[X_1,X_{-1}]|h,w;0,0\rangle\,,\\
&= \langle h,w;0,0|(4L_0+6W_0)|h,w;0,0\rangle = (4h+6w)\,\langle h,w;0,0|h,w;0,0\rangle\,.
\end{aligned}
$$

This implies (assuming $\alpha \in \mathbb{R}$) that $\alpha_{h,w} = \sqrt{4h+6w}$. We can now compute the lowering of this state,

$$
X_1\,|h,w;1,2\rangle = \frac{[X_1,X_{-1}]}{\alpha_{h,w}}\,|h,w;0,0\rangle = \frac{4h+6w}{\alpha_{h,w}}\,|h,w;0,0\rangle = \alpha_{h,w}\,|h,w;0,0\rangle\,. \tag{6.26}
$$

The prefactors, as expected, are related. The generalization to arbitrary descendants is straightforward:

$$
\begin{aligned}
X_{-1}\,|h,w;k,s\rangle &= \alpha_{h+k,w+s}\,|h,w;k+1,s+2\rangle\,,\\
X_{1}\,|h,w;k,s\rangle &= \alpha_{h+k-1,w+s-2}\,|h,w;k-1,s-2\rangle\,.
\end{aligned}
$$

Which is the exact analogue of what happens with J^+, J^- in the SU(2) case, from quantum mechanics textbooks. We thus find

$$
\begin{aligned}
X_{-1}\,|h,w;k,s\rangle &= \alpha_{h+k,w+s}\,|h,w;k+1,s+2\rangle\,, & X_1\,|h,w;k,s\rangle &= \alpha_{h+k-1,w+s-2}\,|h,w;k-1,s-2\rangle\,,\\
Y_{-1}\,|h,w;k,s\rangle &= \gamma_{h+k,w+s}\,|h,w;k+1,s-2\rangle\,, & Y_1\,|h,w;k,s\rangle &= \gamma_{h+k-1,w+s-2}\,|h,w;k-1,s+2\rangle\,,\\
W_{-2}\,|h,w;k,s\rangle &= \beta_{h+k,w+s}\,|h,w;k+2,s\rangle\,, & W_2\,|h,w;k,s\rangle &= \beta_{h+k-2,w+s}\,|h,w;k-2,s\rangle\,.
\end{aligned} \tag{6.27}
$$

The form of each coefficient is computed via recurrence relations. For instance, the recurrence relation and initial condition for $\alpha_{h+k,w+s}$ can be found via inner products

$$
\begin{aligned}
\langle h,w;k,s|X_{-1}^\dagger X_{-1}|h,w;k,s\rangle &= \langle h,w;k,s|X_1X_{-1}|h,w;k,s\rangle = \langle h,w;k,s|X_1^\dagger X_1 + [X_1,X_{-1}]|h,w;k,s\rangle\,,\\
\langle h,w;k,s|Y_{-1}^\dagger Y_{-1}|h,w;k,s\rangle &= \langle h,w;k,s|Y_1Y_{-1}|h,w;k,s\rangle = \langle h,w;k,s|Y_1^\dagger Y_1 + [Y_1,Y_{-1}]|h,w;k,s\rangle\,,\\
\langle h,w;k,s|W_{-2}^\dagger W_{-2}|h,w;k,s\rangle &= -\langle h,w;k,s|W_2W_{-2}|h,w;k,s\rangle = \langle h,w;k,s|W_2^\dagger W_2 - [W_2,W_{-2}]|h,w;k,s\rangle\,,
\end{aligned} \tag{6.28}
$$

which leads to (choosing the coefficients to be real)

$$
\begin{aligned}
\alpha^2_{h+k,w+s} &= \alpha^2_{h+k-1,w+s-2} + 4(h+k) + 6(w+s) \quad \text{subject to} \quad \alpha^2_{h,w} = 4h+6w,\\
\gamma^2_{h+k,w+s} &= \gamma^2_{h+k-1,w+s+2} + 4(h+k) - 6(w+s) \quad \text{subject to} \quad \gamma^2_{h,w} = 4h-6w,\\
\beta^2_{h+k,w+s} &= \beta^2_{h+k-2,w+s} + 16(h+k) \quad \text{subject to} \quad \beta^2_{h,w} = 16h.
\end{aligned} \tag{6.29}
$$

This and the equations for the remaining coefficients are solved by

$$
\alpha^2_{h+k,w+s} = 2(2h+k)(k+1) + \frac{3}{2}(2w+s)(s+2), \tag{6.30a}
$$

$$
\gamma^2_{h+k,w+s} = 2(2h+k)(k+1) + \frac{3}{2}(2w+s)(s-2), \tag{6.30b}
$$

$$
\beta^2_{h+k,w+s} = 4(k+2)(2h+k). \tag{6.30c}
$$

[2] We must make a choice of hermiticity for the operators. We use that which leads to positive norm states for unitarity reasons, in this case corresponding to $X_{\pm1}^\dagger = X_{\mp1}$, $Y_{\pm1}^\dagger = Y_{\mp1}$ (hermitian) and $W_{\pm2}^\dagger = -W_{\mp2}$ (anti-hermitian).

Null states at levels $k = 1, 2, 3$

The next difference with respect to the $\mathfrak{sl}(2,\mathbb{R})$ case comes from the following observation: in this case, for each k-level, there is more than one state. One might be worried that there will be null states in our representation, preventing it from being irreducible, in which case we would then need to quotient them out.

At level $k = 1$ there are two linearly independent states,

$$X_{-1}\left|h,w;0,0\right\rangle \sim \left|h,w,1,2\right\rangle , \quad Y_{-1}\left|h,w;0,0\right\rangle \sim \left|h,w,1,-2\right\rangle . \tag{6.31}$$

At level $k = 2$ we have 5 states. The level $(k,s) = (2,0)$ is degenerate, as there are three combinations of generators that reach it, namely $X_{-1}Y_{-1}\left|h,w;0,0\right\rangle \sim W_{-2}\left|h,w;0,0\right\rangle \sim Y_{-1}X_{-1}\left|h,w;0,0\right\rangle$. Indeed, the Kač determinant for this level is found to be identically zero:

$$\begin{aligned}\det M_{(2,0)} &= \det\begin{pmatrix} \langle XY\,|\,XY\rangle & \langle XY\,|\,YX\rangle & \langle XY\,|\,W\rangle \\ \langle YX\,|\,XY\rangle & \langle YX\,|\,YX\rangle & \langle YX\,|\,W\rangle \\ \langle W\,|\,XY\rangle & \langle W\,|\,YX\rangle & \langle W\,|\,W\rangle \end{pmatrix} \\ &= \det\begin{pmatrix} 4h^2-8h-9w^2+12w & 4h^2-9w^2 & 6w-4h \\ 4h^2-9w^2 & 4h^2-8h-9w^2-12w & 4h+6w \\ 6w-4h & 4h+6w & -4h \end{pmatrix} = 0, \end{aligned} \tag{6.32}$$

The linear combination of these states that is null is found to be

$$\left|\text{null2}\right\rangle = (X_{-1}Y_{-1} - Y_{-1}X_{-1} + 2W_{-2})\left|h,w;0,0\right\rangle = 2(L_{-1}W_{-1} - W_{-1}L_{-1} + W_{-2})\left|h,w;0,0\right\rangle , \tag{6.33}$$

however this provides us with no new information; this is trivially zero because it is precisely the commutator $[W_{-1}, L_{-1}] = W_{-2}$. Hence this is not a real null state, it is the zero state.

We are going to conjecture that the same thing will happen at each and every k-level in the representation, there will be no real null states. To support this statement we will now compute the null states in the next level and ensure that it is the case again. First, notice that there will be two trivial states coming from X_{-1}, Y_{-1} acting on $\left|\text{null2}\right\rangle$, the only non trivial one is

$$\left|\text{null3}\right\rangle = (Y_{-1}X_{-1}^2 + X_{-1}^2Y_{-1} - 2X_{-1}Y_{-1}X_{-1})\left|h,w;0,0\right\rangle . \tag{6.34}$$

But this expression, as promised, gives us no new information, as if we commute the Y's past one X, we find that $\left|\text{null3}\right\rangle$ is trivially zero.

This is further support for our claim about the representation being irreducible, with well defined quantum numbers $\left|h,w;k,s\right\rangle$, a claim that we will assume to be true from now on.

Something we must consider before moving on is the space of states comprising the representation, as we will have to sum over them later. Not only that, some of them will be degenerate, with degeneracy 2, due to the algebra. They are shown in Figure 6.1

6.3.2 Boundary states and inner product

In the previous chapter we found states with the appropriate transformation law that we required in analogy with the classical $U(x)$ field. For the higher spin generalization we are seeking the exact same kind of states, satisfying

$$G(L)\bar{G}(R^{-1})\left|\Sigma\right\rangle = \left|L\Sigma R\right\rangle , \tag{6.35}$$

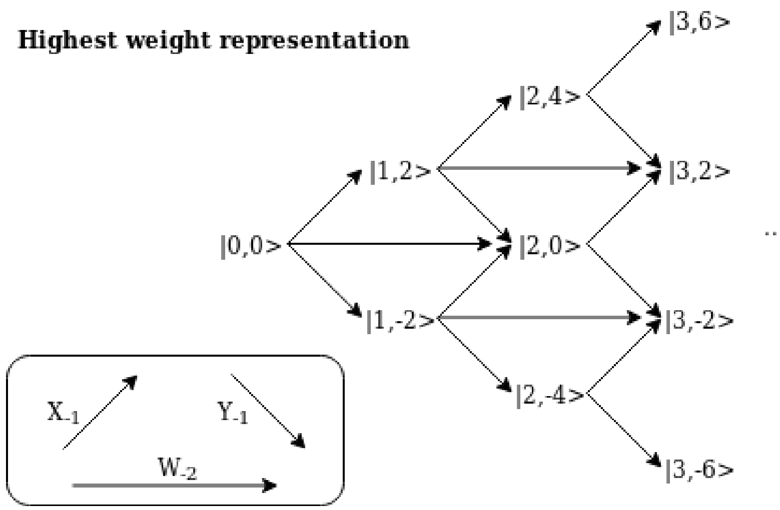

Figure 6.1: Tower of descendants built from the highest weight state $|h, w; 0, 0\rangle$ with the raising operators. We can clearly see that not all pairs (k, s) are allowed. It is also clear that the states lying "inside" the tower are degenerate.

but for $\mathrm{SL}(N, \mathbb{R})$, therefore we want to solve the following equations:

$$Q_a^{\mathrm{Ish}(s)} |\Sigma_{\mathrm{Ish}}\rangle = \left(W_a^{(s)} - (-1)^s \overline{W}_{-a}^{(s)} \right) |\Sigma_{\mathrm{Ish}}\rangle = 0, \tag{6.36a}$$

$$Q_a^{\mathrm{cross}(s)} |\Sigma_{\mathrm{cross}}\rangle = \left(W_a^{(s)} - (-1)^a (-1)^s \overline{W}_{-a}^{(s)} \right) |\Sigma_{\mathrm{cross}}\rangle = 0, \tag{6.36b}$$

where $2 \leq s \leq N$, $a \leq |s-1|$. The s label is the conformal dimension of the operators. For $N = 3$, we have a set of 8 equations per case, and we can make contact with the usual notation of the generators identifying $W_a^{(2)} \equiv L_a$, $W_a^{(3)} \equiv W_a$. Note that four of those equations, $Q_0^{(2)}, Q_0^{(3)}, Q_{\pm 2}^{(3)} = 0$, are the same in both cases.

We will postulate the solution to be the linear combination of highest weight states belonging to $\mathfrak{sl}(3, \mathbb{R}) \times \mathfrak{sl}(3, \mathbb{R})$,

$$|\Sigma\rangle = \sum_{k, \bar{k}, s, \bar{s}} \Phi(k, \bar{k}, s, \bar{s}) \, |h, w; k, s\rangle \otimes \overline{|h, w; k, s\rangle} \tag{6.37}$$

(where the dependence of Φ on h, w and barred ones has been omitted and does not affect the result, only the normalization). Notice that we have purposely omitted the range of the sums, however they are important as the pairs of values (k, s) are constrained (Figure 6.1). The correct description of the range of the sums is generated by

$$\sum_{k', s'} |k', s'\rangle = \sum_{k=0}^{\infty} \left(\sum_{s=-k}^{k} d_{2k, 4s} \, |2k, 4s\rangle + \sum_{s=-k}^{k+1} d_{2k+1, 4s-2} \, |2k+1, 4s-2\rangle \right), \tag{6.38}$$

where $d_{k,s}$ indicates the degeneracy of the $|k, s\rangle$ state; 1 for the states bordering the tower and 2 for the states inside. This will be the correct range of the sums.

Once the ranges have been addressed, solving the equations just requires a good deal of book-keeping. A wise choice is to begin by solving $Q_0^{(2)}, Q_0^{(3)} = 0$, which relate the highest weights as

$h = \bar{h}$, $w = -\bar{w}$ for both cases. Then solving $Q^{(3)}_{\pm 2} = 0$ is also general for both cases. After that, one has to solve separately the remaining four equations, which is easier if they are combined into equations for $X_\pm, Y_\pm$ (6.23) rather than $L_\pm, W_\pm$. The final result is[3]

$$|\Sigma_{\text{Ish}}\rangle = P \sum_{k=0}^{\infty} \left(\sum_{s=-k}^{k} |2k, 2k, 4s, -4s\rangle + \sum_{s=-k}^{k+1} |2k+1, 2k+1, 4s-2, -4s+2\rangle \right), \tag{6.39a}$$

$$|\Sigma_{\text{cross}}\rangle = P \sum_{k=0}^{\infty} \left(\sum_{s=-k}^{k} |2k, 2k, 4s, -4s\rangle - \sum_{s=-k}^{k+1} |2k+1, 2k+1, 4s-2, -4s+2\rangle \right), \tag{6.39b}$$

where P is just an overall normalization. The group elements associated with the adjoint actions leading to each equation, are found to be

$$\Sigma_{\text{Ish}}\, \overline{W}^{(s)}_a\, \Sigma^{-1}_{\text{Ish}} = -\overline{W}^{(s)}_a, \quad \Sigma_{\text{Ish}} = \exp\left\{ i\frac{\pi}{2}(L_1 - L_{-1}) \right\} = \exp\left\{ \frac{\pi}{2}(W_1 + W_{-1}) \right\}, \tag{6.40a}$$

$$\Sigma_{\text{cross}}\, \overline{W}^{(s)}_a\, \Sigma^{-1}_{\text{cross}} = -(-1)^a \overline{W}^{(s)}_a, \quad \Sigma_{\text{cross}} = \exp\left\{ \frac{\pi}{2}(L_1 + L_{-1}) \right\} = \exp\left\{ i\frac{\pi}{2}(W_1 - W_{-1}) \right\}. \tag{6.40b}$$

Inner product

The inner product will be computed as in the SL(2, $\mathbb{R}$) case (5.34), using the transformation property of $|\Sigma\rangle$ (6.35). The main difference is that now we have $M = L\Sigma R\Sigma^{-1} = V \exp(-\alpha L_0 - \beta W_0) V^{-1}$, with $\alpha = \alpha(x_i, x_f)$ and $\beta = \beta(x_i, x_f)$:

$$\begin{aligned} W_{\mathcal{R}}(x_i, x_f) &= \langle \Sigma | G(L) \overline{G}(R^{-1}) \Sigma \rangle = \langle \Sigma | L \Sigma R \rangle = \langle \Sigma | G(M) | \Sigma \rangle \\ &= P^2 \sum_{k=0}^{\infty} \left[\sum_{s=-k}^{k} d_{2k,4s} e^{-\alpha(h+2k)} e^{-\beta(w+4s)} + \sum_{s=-k}^{k+1} d_{2k+1,4s-2} e^{-\alpha(h+2k+1)} e^{-\beta(w+4s-2)} \right] \\ &= P^2 \frac{e^{-\alpha h - \beta w}}{\left(1 - e^{-2\alpha}\right)\left(e^{2\beta} - e^{-\alpha}\right)\left(e^{-2\beta} - e^{-\alpha}\right)}. \end{aligned} \tag{6.41}$$

From now on, we will drop the normalization P^2. Some comments are in order: first of all, what we just computed is an SL(3, $\mathbb{R}$) character of $G(L\Sigma R\Sigma^{-1})$. This result is in accordance with usual CFT knowledge. In the Virasoro algebra the characters are computed as

$$\chi^{\text{Vir}}(q) = q^{h-c/12} \prod_{n}^{\infty} \frac{1}{(1-q^n)}, \quad q = e^{2\pi i}. \tag{6.42}$$

This function is counting the number of independent raising operators $L_{-1}, L_{-2}, \ldots$, which in the Virasoro case happens to be infinite. In the present case, we only have three, each of them accounting for each factor in the denominator[4].

[3] There's a caveat: if one plugs any of these states (Ishibashi/crosscap) in their defining equations, one will not find them to be satisfied: as a matter of fact, some states don't appear to vanish. If one tracks their origin, the explanation arises: the equations involve X_1, Y_1, W_2 operators acting on the whole tower of states, hence when they act on certain states bordering the tower Figure 6.1, the resulting state will lie outside the representation (e.g. $X_1 |1, -2\rangle \sim |0, -4\rangle$). However these states are artifacts of the notation used ($|0, -4\rangle \sim X_1 Y_{-1} |0, 0\rangle = [X_1, Y_{-1}] |0, 0\rangle = 0$).

[4] I would like to thank Jan de Boer for his insightful comments on the matter.

The exponents of each of the three factors are determined by the action of each generator: W_{-2} raises the L_0 eigenvalue ("coupled" to α) by 2, $\left(1-e^{-2\alpha}\right)$; X_{-1} raises the L_0 eigenvalue by 1 and the W_0 eigenvalue ("coupled" to β) by 2, $\left(e^{2\beta}-e^{-\alpha}\right)$; Y_{-1} raises the L_0 eigenvalue by 1 and lowers the W_0 eigenvalue by 2, $\left(e^{-2\beta}-e^{-\alpha}\right)$.

It will be interesting for us to compute the $\beta \to 0$ limit of this result:

$$\lim_{\beta\to 0} W_{\mathcal{R}}(x_i, x_f) = \frac{e^{-\alpha h}}{(1-e^{-\alpha})^3}. \tag{6.43}$$

Note the similarity with the result found in [4], although not being identical.

In principle the procedure followed would work for any SL$(N,\mathbb{R})$. The steps one has to follow are the same: find a basis of generators diagonalizing the Cartan subalgebra, check whether there are null states in the spectrum, if not a generalization of (6.37) can be used as ansatz for (6.36a) and its solution can be used to compute the Wilson line as an inner product. The derivation for the SL$(4,\mathbb{R})$ case can be found in Appendix D, which results in

$$W_{\mathcal{R}}(x_i, x_f) = \frac{e^{-\alpha h-\beta w-\gamma u}}{\left(e^{3\gamma}-e^{-\alpha}\right)\left(e^{-2\beta-2\gamma}-e^{-\alpha}\right)\left(e^{2\beta-2\gamma}-e^{-\alpha}\right)\left(e^{-2\beta+\gamma}-e^{-2\alpha}\right)\left(e^{2\beta+\gamma}-e^{-2\alpha}\right)\left(e^{-\gamma}-e^{-3\alpha}\right)}. \tag{6.44}$$

It would be interesting to use this result, however the literature on SL$(4,\mathbb{R})$ gravity is scarce, with the only reference (at the time of writing this thesis) being [44]. It is not obvious how to conjecture from the SL$(2,\mathbb{R})$, SL$(3,\mathbb{R})$, SL$(4,\mathbb{R})$ cases a result for SL$(N,\mathbb{R})$, however it would be very interesting to find it and perhaps even a hs$[\lambda]$ generalization, which would be a step closer to Vasiliev's higher spin theory.

The diagonal embedding truncation

As we have discussed in Chapter 4, the diagonal embedding black hole is an SL$(2,\mathbb{R})\times$U(1) truncation of SL$(3,\mathbb{R})$. In this case we will have generators $J_{0,\pm1} = \{\frac{1}{2}L_0, \pm\frac{1}{4}W_{\pm2}\}$ for the $\mathfrak{sl}(2,\mathbb{R})$ part and W_0 for the $\mathfrak{u}(1)$. The fact that it is a direct product can be seen (at the level of the algebra) from their commutation relations, and allows us to easily construct the tower of states $|h,w;k,s\rangle$ as

$$|h,w;0,0\rangle \xrightarrow{J_{-1}} |h,w;1,0\rangle \xrightarrow{J_{-1}} |h,w;2,0\rangle \xrightarrow{J_{-1}} |h,w;3,0\rangle \ldots$$

These states can then be written as $|h,w;k,0\rangle \equiv |h,k\rangle \otimes |w\rangle$.

The representation within the Wilson line is that of two copies of this truncation, therefore the states we consider are $|h,k\rangle\,|w\rangle \otimes \overline{|h,k\rangle\,|w\rangle}$. We would follow the same procedure as above to solve the Ishibashi/crosscap equations, which for the $\mathfrak{sl}(2,\mathbb{R})$ part it is the same as in [4] and the $\mathfrak{u}(1)$ part is trivial as it only involves

$$(W_{\pm2} + \overline{W}_{\mp2})\,|\Sigma\rangle = 0, \tag{6.45}$$

therefore it only requires $\bar{w} = -w$. The resulting states are (up to a normalization P)

$$|\Sigma_{\text{Ish}}\rangle = \sum_{k=0}^{\infty} |h,k\rangle\,|w\rangle \otimes |h,k\rangle\,|-w\rangle\,, \tag{6.46a}$$

$$|\Sigma_{\text{cross}}\rangle = \sum_{k=0}^{\infty} (-1)^k\,|h,k\rangle\,|w\rangle \otimes |h,k\rangle\,|-w\rangle\,. \tag{6.46b}$$

Computing the inner product of any of them with the group element $G(M)$ alluded above leads to

$$W_{\mathcal{R}} = \langle \Sigma | G(L\Sigma R\Sigma^{-1}) | \Sigma \rangle = \sum_{k=0}^{\infty} e^{-\alpha(h+k)-\beta w} = \frac{e^{-\alpha h}}{1-e^{\alpha}}\, e^{-\beta w}. \tag{6.47}$$

This is what we expected as the truncation is a direct product $\mathrm{SL}(2,\mathbb{R}) \times U$, the $\mathfrak{sl}(2,\mathbb{R})$ character times a phase due to the holonomy of the $\mathfrak{u}(1)$ connections.

6.3.3 Relation to the probe's Casimirs

Before delving into specific calculations, develop a deeper understanding of the quantities appearing in (6.41). In particular, we want to connect the coefficients α and β to the probe's charges, which are given by the Casimirs c_2, c_3.

In the previous section we reviewed the semiclassical computation from [2], which only required the $\mathrm{SL}(3,\mathbb{R})$ Casimir elements c_2, c_3 in the semiclassical (large h, w) limit. However, as they correctly point out, in a proper quantum treatment they must be computed through normal ordering, acting on any state of the representation. Using the conventions from Appendix B, we have

1. Quadratic Casimir:

$$C_2 = \delta^{ab} T_a T_b = \frac{1}{2} L_0^2 - \frac{1}{4} L_{(1} L_{-1)} + \frac{3}{8} W_0^2 - \frac{1}{4} W_{(1} W_{-1)} + \frac{1}{16} W_{(2} W_{-2)}, \tag{6.48}$$

where the subindices between parenthesis are symmetrized, $T_a = \{L_n, W_m\}$, and the Killing form is defined as $\delta_{ab} = \mathrm{Tr}_f(T_a T_b)$, and δ^{ab} is its inverse. When evaluated on the highest weight state (without loss of generality), we find after normal ordering

$$c_2 = \langle h, w | C_2 | h, w \rangle = \frac{1}{2} h^2 - 2h - \frac{3}{8} w^2. \tag{6.49}$$

This result indeed agrees with the large h, w limit of [2], where the second term is dismissed as it is subleading.

2. Cubic Casimir:

$$C_3 = h_{abc} T^a T^b T^c = h_{abc} \delta^{ad} \delta^{be} \delta^{cf} T_d T_e T_f, \tag{6.50}$$

where $h_{abc} = \mathrm{Tr}_f(T_{(a} T_b T_{c)})$. In this case there are too many terms as for displaying them here. Its evaluation yields

$$c_3 = \langle h, w | C_3 | h, w \rangle = \frac{3}{8} h^2 w - \frac{3}{32} w^3 - \frac{3}{4} h w + \frac{3}{4} w. \tag{6.51}$$

Later on we will be interested in matching our result to those of [2]. This means that the $w = 0$ case is of special interest, as we can see that it implies the vanishing of the cubic Casimir.

Now that we found the appropriate Casimirs, we can ask about their relation to the conjugacy class labels α, β. In [2, 32] this conjugacy class is constructed out of an element of the algebra P_0, as

$$M = V \exp(\mathbb{P}) V^{-1} = V \exp(-\alpha L_0 - \beta W_0) V^{-1}, \tag{6.52}$$

where $\mathbb{P}$ is the $\mathfrak{sl}(3,\mathbb{R})$ truncation of (6.12)

$$\mathbb{P} = -2\Delta\alpha_2\, P_0 - 3\Delta\alpha_3\,(P_0 \times P_0), \quad P_0 \times P_0 = h_{abc} P_0^a P_0^b T_c. \tag{6.53}$$

We want to connect these two unknowns, $\Delta\alpha_2, \Delta\alpha_3$ to α, β such that we get a better understanding of the probe. This relation is mediated by the Casimirs, which is why we needed to deal with them above.

The way to begin this task is by finding P_0. There are two constraints on it, namely the quadratic and cubic traces (6.6) are imposed to be

$$\mathrm{Tr} P_0^2 \equiv \delta_{ab} P_0^a P_0^b = c_2, \quad \mathrm{Tr} P_0^3 = h_{abc} P_0^a P_0^b P_0^c = c_3. \tag{6.54}$$

Using the ansatz $P_0 = \lambda L_0 + \xi W_0$, a solution to these constraints is

$$\lambda = \frac{1}{4\sqrt{3}}\sqrt{-\frac{6c_2^2\sqrt[3]{6}}{\Delta^2} + 12c_2 - 6^{2/3}\Delta^2}, \quad \xi = \frac{6^{2/3}c_2 + \sqrt[3]{6}\Delta^2}{8\Delta}, \tag{6.55}$$

where we have defined

$$\Delta \equiv \left(\sqrt{6(6c_3^2 - c_2^3)} - 6c_3\right)^{1/3}. \tag{6.56}$$

This solution, though is not unique, however it is the simplest one.

We can now compute (6.53) and solve (6.52)

$$\alpha \equiv 2\Delta\alpha_2\lambda + 4\Delta\alpha_3\xi\lambda, \quad \beta \equiv 2\Delta\alpha_2\xi + \Delta\alpha_3\left(\frac{3\lambda^2}{2} - 2\xi^2\right). \tag{6.57}$$

This allows us to establish a relation between the Casimirs, which measure the strength of the charges carried by the probe, and the conjugacy class labels, which are equipped with some knowledge of the geometry. In fact, if we follow the same procedure for a probe carrying no higher spin (i.e. $c_3 = 0$)[5], we find

$$P_0(c_3 = 0) = \sqrt{\frac{c_2}{2}} L_0, \quad \alpha(c_3 = 0) = \Delta\alpha_2\sqrt{2c_2}, \quad \beta(c_3 = 0) = \frac{3\Delta\alpha_3 c_2}{4}. \tag{6.58}$$

We can see that both P_0 and α match precisely those of the $SL(2,\mathbb{R})$ case. This is what we expected, as we are considering now a probe with no higher spin charge, as in [2]. In fact we can see from here that if we not only consider a probe without higher spin charge, but also embedded in a pure gravity geometry, we will need $\Delta\alpha_3 = 0$, as in that case $\beta = 0$. Besides this remark, the actual interpretation of $\Delta\alpha_3$ remains obscure after this analysis.

6.3.4 Evaluation of the Wilson line: match with previous results

After the preliminaries above, now it is time to use (6.41) to generate some results. We will begin by matching this formalism with results from [2, 4]. This will be more a consistency check rather than novel results, but it is a necessary step before moving on. However, even at the semiclassical

[5] It is actually better to compute it by plugging $c_3 = 0$ in the traces of P_0 (6.54) than in the general solution (6.55), due to the amount of square and cubic roots involved.

level, the Wilson lines obtained will actually be a bit more general than the ones computed in [2] as their focus was on computing entanglement entropies, therefore their probes carried no higher spin charge and were computed on a time slice.

For this purpose we need to compute

$$g_L(x_f)g_L(x_i)^{-1}\tilde{g}_R(x_i)^{-1}\tilde{g}_R(x_f) = V\exp\{-\alpha(x_i,x_f)\,L_0 - \beta(x_i,x_f)\,W_0\}V^{-1}, \tag{6.59}$$

which will determine α, β as functions of the coordinates, as it did in the $\mathrm{SL}(2,\mathbb{R})$ case relating α and the geodesic distance.

Geodesic distance in pure gravity case

Had we been given a set of connections $A, \bar{A} \in \mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{sl}(3,\mathbb{R})$, the values of α and β would be identically the same as for the fully-fledged $\mathrm{SL}(2,\mathbb{R})$ group studied in the last section (meaning for instance $\beta = 0$). This is expected because in the fundamental representation of $\mathrm{SL}(3,\mathbb{R})$, the generators corresponding to its $\mathrm{SL}(2,\mathbb{R})$ subgroup are those of the adjoint representation of the $\mathrm{SL}(2,\mathbb{R})$ group. But note that (6.59) is representation independent, therefore both inner products must be the same.

We will explicitly check this is the case. Consider the AAdS_3 group elements for constant $C = k\mathcal{L}/2\pi$,

$$g_L(x) = b(\rho)^{-1}\, e^{-(L_1 - CL_{-1})\,x^+}, \quad \tilde{g}_R(x) = e^{-(L_{-1} - CL_1)\,x^-}\, b(\rho)^{-1}. \tag{6.60}$$

where $x^\pm = t \pm \phi$. Using (6.59) in the fundamental representation with some patience and a good amount of hyperbolic identities, we find

$$\cosh(\alpha) = \frac{1}{2}\sigma(x_i, x_f)^2 - 1, \tag{6.61}$$

where $\sigma(x_i, x_f)$ is the same as in (5.49). As argued above, it would make sense that $\alpha_{\mathrm{SL}(2,\mathbb{R})} = \alpha_{\mathrm{SL}(3,\mathbb{R})}$, but at first glance that doesn't seem to be the case, if one compares this equation to [4].

Luckily, hyperbolic identities come to our rescue, as

$$2\,\mathrm{arcosh}(x) = \mathrm{arcosh}(2x^2 - 1), \quad x \geq 1, \tag{6.62}$$

where in our case $\sigma = 2x$, finding an exact match (also, both terms $\cosh(x-y)\cosh(x+y)$ and $-\sinh(x-y)\sinh(x+y)$ are bigger than zero, and the factors before are positive and exponential-like therefore it is not hard to see that $s \geq 2$ as required).

Let us now consider the case of an open Wilson line stretching between two different points of the AdS boundary. We will first describe it for the $\mathrm{SL}(2,\mathbb{R})$ subgroup, to exemplify the computation. After that we will deal with the diagonal and principal embedding black holes.

Global AdS_3

We could directly use the above result for α, (6.61), with $C = -1/4$, plugging it in the inner product and expanding to leading order. However, as a warm up for the next cases where we don't have a close form for α, we will choose not to follow that path. Instead, let us compute (6.59) for $\beta = 0$. Tracing said equation leaves us with

$$\frac{\cos(\Delta t) - \cos(\Delta\phi)}{8\epsilon^2} + (\cos(\Delta t) + \cos(\Delta\phi)) + 2\epsilon^2(\cos(\Delta t) - \cos(\Delta\phi)) = 2\cosh(\alpha), \tag{6.63}$$

where $e^{-\rho_i} = e^{-\rho_f} = \epsilon$, introducing our UV cutoff $\epsilon \to 0$. Solving for a, and substituting it in the expression of the propagator (6.43), we find that the leading divergent piece is

$$W_{\mathcal{R}}^{\text{empty,spin}-3,b\to 0} = \langle\Sigma|G(e^{-\int A})\overline{G}(e^{-\int \bar{A}})|\Sigma\rangle = \frac{e^{-\alpha h}}{(1-e^{-\alpha})^3} = \left(\frac{8\epsilon^2}{\cos(\Delta t)-\cos(\Delta\phi)}\right)^{2h} + O(\epsilon^{2h}). \quad (6.64)$$

As expected, this result is the same to the one computed via (6.61) or with the full-fledged SL(2, ℝ) group elements (not the SL(2, ℝ) ⊂ SL(3, ℝ) subgroup), i.e.

$$W_{\mathcal{R}}^{\text{empty,spin}-2} = \langle\Sigma|G(e^{-\int A})\overline{G}(e^{-\int \bar{A}})|\Sigma\rangle = \frac{e^{-\alpha h}}{1-e^{-\alpha}} = \left(\frac{8\epsilon^2}{\cos(\Delta t)-\cos(\Delta\phi)}\right)^{2h} + O(\epsilon^{2h}). \quad (6.65)$$

Diagonal embedding black hole

We begin the higher spin study with the diagonal embedding black hole, as it is simple enough to compute things without too much trouble. It will have as group elements

$$\begin{aligned} g_L(x) &= b(\rho)^{-1}\, e^{-(W_2+\omega\, W_{-2}-q\, W_0)\, x^+ - \eta\, W_0\, x^-/2}, \\ \tilde{g}_R(x) &= e^{(W_{-2}+\omega\, W_2 - q\, W_0)\, x^- + \eta\, W_0\, x^+/2}\, b(\rho)^{-1}. \end{aligned} \quad (6.66)$$

In this case, we need at least two equations to fix α, β. As before, the most straightforward one is found by tracing both sides of the equation, although we will leave this one for later. It is better to first diagonalize $g_L(x_f)g_L(x_i)^{-1}\tilde{g}_R(x_i)^{-1}\tilde{g}_R(x_f)$. By doing so we can compute β, matching the eigenvalues of both sides of (6.59):

$$\exp\left(\frac{4\beta}{3}\right) = \exp\left[\frac{2}{3}(x_f^- - x_f^+ - x_i^- + x_i^+)(\eta+2q)\right] \quad \Rightarrow \quad \beta = -(\eta+2q)\Delta\phi. \quad (6.67)$$

Introducing this on the trace of equation (6.59) we find

$$\begin{aligned} \cosh(\alpha) =& \cosh(2\Delta\rho)\cosh\left(4\sqrt{\omega}(\Delta t - \Delta\phi)\right)\cosh\left(4\sqrt{\omega}(\Delta t + \Delta\phi)\right) \\ &- \frac{1}{2}\left(\omega e^{-2\rho_f - 2\rho_i} + \frac{e^{2\rho_f+2\rho_i}}{\omega}\right)\sinh\left(4\sqrt{\omega}(\Delta t - \Delta\phi)\right)\sinh\left(4\sqrt{\omega}(\Delta t + \Delta\phi)\right) \end{aligned} \quad (6.68)$$

At this point it is worth elaborating on the interpretation of α, β for the case at hand: a rather interesting feature is the fact that β doesn't depend on the cutoff, it will give a finite contribution to the propagator. This is related to the fact that the diagonal embedding black hole can be regarded as a consistent truncation of SL(3, ℝ) as SL(2, ℝ)×U(1), where the Wilson line for the U(1) piece is yielding a phase in the full propagator. This behaviour was already noticed in [40]. By the same reasoning, α should account for the SL(2, ℝ) part, which is why the form of (6.68) is similar to (5.49) (with $\sigma = \cosh(\alpha/2)$), meaning that in the diagonal embedding α is still the usual geodesic distance.

We can easily solve for α to leading divergent order, and plug that solution in the propagator (6.41) with $h \to 2h$, with this extra factor of 2 coming from the difference between $2L_0$ of the diagonal embedding and J_0 of the $\mathfrak{sl}(2, \mathbb{R})$ subalgebra (which is related to the conformal weight, see subsection B.2.1).

$$W_{\mathcal{R}}^{\text{diagBH}} = \langle\Sigma|G(e^{-\int A})\overline{G}(e^{-\int \bar{A}})|\Sigma\rangle \simeq (2\omega)^{2h} e^{\Delta\phi(\eta+2q)w}\left(\frac{\epsilon^4}{\cosh\left(8\Delta\phi\sqrt{\omega}\right)-\cosh\left(8\Delta t\sqrt{\omega}\right)}\right)^{2h}. \quad (6.69)$$

We can now make contact with [2], by taking suitable limits: they consider a particle with no $U(1)$ charge ($w \to 0$) and, for the purpose of computing Entanglement Entropy, they have to consider a constant time slice ($\Delta t \to 0$),

$$W_{\mathcal{R}}^{\text{diagBH}} \to (2\omega)^{2h} \left(\frac{\epsilon^4}{\cosh\left(8\Delta\phi\sqrt{\omega}\right) - 1} \right)^{2h} = \left(\frac{\omega\epsilon^4}{\sinh^2\left(4\Delta\phi\sqrt{\omega}\right)} \right)^{2h} . \tag{6.70}$$

It is nice to check that when we use their prescription to compute the entanglement entropy, we find an exact match

$$S_{\text{EE}} = -\log W_{\mathcal{R}}^{\text{diagBH}} = 2h \log\left(\frac{1}{\sqrt{\omega}\epsilon^2} \sinh\left(4\Delta\phi\sqrt{\omega}\right)\right). \tag{6.71}$$

Principal embedding black hole

In the principal embedding black hole case, a different approach must be taken, as exponentiating the connections right away is not feasible. For simplicity, we will choose one of the endpoints to be located at $x_i = (0,0)$.

If one tries to compute the exponential of the connections, nothing sensible will come out of that computation. In order to make progress, we will first use the results shown in Appendix C of [2]. There, they manage to give "simple" expressions for the eigenvalues and eigenvectors of the angular part of the connections, namely

$$\lambda_\phi = 2\sqrt{\frac{2\pi\mathcal{L}}{k}} \operatorname{diag}\left(\frac{3 + C\left(\sqrt{4C-3} - 2\right)}{\sqrt{C}(2C-3)}, \frac{2}{\sqrt{C}}, \frac{3 - C\left(2 + \sqrt{4C-3}\right)}{\sqrt{C}(2C-3)} \right), \tag{6.72}$$

$$V_a = \begin{pmatrix} \frac{2}{C}\left(C - 1 - \sqrt{4C-3}\right) & \frac{2}{C}(2-C) & \frac{2}{C}\left(C - 1 + \sqrt{4C-3}\right) \\ \left(\sqrt{4C-3} - 1\right)\sqrt{\frac{k}{2\pi\mathcal{L}C}} & 2\sqrt{\frac{k}{2\pi\mathcal{L}C}} & \left(-\sqrt{4C-3} - 1\right)\sqrt{\frac{k}{2\pi\mathcal{L}C}} \\ \frac{k}{2\pi\mathcal{L}} & \frac{k}{2\pi\mathcal{L}} & \frac{k}{2\pi\mathcal{L}}, \end{pmatrix} \tag{6.73}$$

with $a_\phi = V_a \lambda_\phi V_a^{-1}$. The corresponding ones for the barred connection can be found to be

$$\bar{\lambda}_\phi = -\lambda_\phi, \quad V_{\bar{a}} = g V_a, \quad g = \begin{pmatrix} 0 & 0 & -2 \\ 0 & -1 & 0 \\ -1/2 & 0 & 0 \end{pmatrix}. \tag{6.74}$$

This information is enough for computing entanglement entropy as it is restricted to a time slice, however we are here interested in the full connections, therefore we need to include their time components. However we can be clever about it: the connections, being on-shell and constant, need to satisfy $[a_+, a_-] = 0$ (and the same for the barred ones), as can be checked explicitly. This further implies $[a_\phi, a_t] = 0$, which means that the time and angular part of the connections can be diagonalized simultaneously, i.e. they have the same eigenvectors. Then, the only thing left is finding the eigenvalues of a_t, which is straigthforward given its eigenvectors:

$$\lambda_t = \frac{2(C-3)x}{(2C-3)\sqrt{C}} \frac{2\pi\mathcal{L}}{k} \operatorname{diag}(1, 0, -1) \tag{6.75}$$

and in this case, we have $\bar{\lambda}_t = \lambda_t$. Therefore, the group elements can be decomposed as

$$g_L(x) = b(\rho)^{-1} \left(V_a e^{-\lambda_t t - \lambda_\phi \phi} V_a^{-1}\right), \quad g_R(x) = \left(g V_a e^{\lambda_t t - \lambda_\phi \phi} V_a^{-1} g^{-1}\right) b(\rho)^{-1}. \tag{6.76}$$

Borrowing the notation from [2], we can compute m_1 and m_2, which precisely match those corresponding to entanglement entropy in the suitable limit ($t = 0$, $\lambda_\phi \to -\lambda_\phi$; that last one because of notational difference).

The next step is finding the eigenvalues of M, in order to match one with $\beta(t, \phi)$, as in the diagonal embedding case. To leading order, this can be done as above, finding

$$e^{4\beta/3} = \frac{m_2}{m_1}. \tag{6.77}$$

Although the value of β here is independent of the cutoff, it is unclear at this point the finiteness of this contribution to the propagator, as this is to leading order in ϵ. We can plug this is the equation of the trace of (6.59), solve it to leading divergent order in ϵ (it is actually more convenient to solve for e^α instead of α itself).

$$e^{-\alpha - \frac{2\beta}{3}} + e^{\alpha - \frac{2\beta}{3}} + e^{\frac{4\beta}{3}} = \mathrm{Tr}(M) \Rightarrow e^\alpha = \frac{m_1 \sqrt{e^{4\beta/3}}}{\epsilon^4} + O(\epsilon^{-2}) = \frac{\sqrt{m_1 m_2}}{\epsilon^4} + O(\epsilon^{-2}). \tag{6.78}$$

Inserting this in the expression for the propagator (6.41), we find

$$W_{\mathcal{R}}^{\mathrm{ppalBH}} = \langle \Sigma | G(e^{-\int A}) \overline{G}(e^{-\int \bar{A}}) | \Sigma \rangle \simeq \left(\frac{m_2}{m_1}\right)^{-\frac{3w}{4}} \left(\frac{\sqrt{m_1 m_2}}{\epsilon^4}\right)^{-h}, \tag{6.79}$$

which is not surprising as it obviously matches the expected behaviour in the equal time, $w = 0$ case, however in a more general case, the values of m_1 and m_2 are

$$\begin{aligned} m_1 = &\left(\frac{C e^{-t\lambda_\tau}}{4(C-3)x}\right)^2 \left((x-3)e^{(\lambda_1+\lambda_2)\phi + 2t\lambda_\tau} - 2x e^{t\lambda_\tau - \lambda_2 \phi} + (x+3)e^{-\lambda_1 \phi}\right) \\ &\times \left(-2x e^{t\lambda_\tau - \lambda_2\phi} + (x+3) e^{2t\lambda_\tau - \lambda_1 \phi} + (x-3) e^{(\lambda_1+\lambda_2)\phi}\right) \left(\frac{k}{2\pi\mathcal{L}}\right)^2, \end{aligned} \tag{6.80}$$

and

$$
\begin{aligned}
m_2 = &\left(\frac{Ce^{-2t\lambda_\tau}}{16\sqrt{2}(C-3)^2x^2}\right)^2 \Big[\Big(\left(-x^2+2x+3\right)e^{(\lambda_1+\lambda_2)\phi+2t\lambda_\tau} - 4xe^{t\lambda_\tau-\lambda_2\phi} + \left(x^2+2x-3\right)e^{\lambda_1(-\phi)}\Big) \\
&\times \Big(\left(x^2+2x-3\right)e^{2t\lambda_\tau-\lambda_1\phi} - 4xe^{t\lambda_\tau-\lambda_2\phi} + \left(-x^2+2x+3\right)e^{(\lambda_1+\lambda_2)\phi}\Big) \\
&\times \Big((-2C+x+3)e^{(\lambda_1+\lambda_2)\phi+2t\lambda_\tau} + (2C+x-3)e^{\lambda_1(-\phi)} - 2xe^{t\lambda_\tau-\lambda_2\phi}\Big) \\
&\times \Big((2C+x-3)e^{2t\lambda_\tau-\lambda_1\phi} + (-2C+x+3)e^{(\lambda_1+\lambda_2)\phi} - 2xe^{t\lambda_\tau-\lambda_2\phi}\Big) \\
&- \Big((x-3)e^{(\lambda_1+\lambda_2)\phi+2t\lambda_\tau} - 2xe^{t\lambda_\tau-\lambda_2\phi} + (x+3)e^{\lambda_1(-\phi)}\Big) \\
&\times \Big(-2xe^{t\lambda_\tau-\lambda_2\phi} + (x+3)e^{2t\lambda_\tau-\lambda_1\phi} + (x-3)e^{(\lambda_1+\lambda_2)\phi}\Big) \\
&\times \Big(2(C-2)xe^{t\lambda_\tau-\lambda_2\phi} + (x-3)(C+x-1)e^{(\lambda_1+\lambda_2)\phi+2t\lambda_\tau} + (x+3)(C-x-1)e^{\lambda_1(-\phi)}\Big) \\
&\times \Big(C\left((x-1)e^{2t\lambda_\tau-\lambda_1\phi} + 2xe^{t\lambda_\tau-\lambda_2\phi} + (x+1)e^{(\lambda_1+\lambda_2)\phi}\right) \\
&-4x\left(e^{(\lambda_1+\lambda_2)\phi} + e^{t\lambda_\tau-\lambda_2\phi} + e^{2t\lambda_\tau-\lambda_1\phi}\right)\Big)\Big]\left(\frac{k}{2\pi\mathcal{L}}\right)^2,
\end{aligned} \tag{6.81}
$$

where we have $\lambda_\phi = \mathrm{diag}(\lambda_1, \lambda_2, -\lambda_1 - \lambda_2)$ and $\lambda_t = \mathrm{diag}(\lambda_\tau, 0, -\lambda_\tau)$.

It is important to notice that in all of the examples we have discussed, there are no corrections coming from the denominator of the inner product (6.41). The reason for this is that they have been obscured by the cutoff, as we have worked to leading order in it.

Chapter 7

Conclusion and outlook

Three dimensional gravity was initially thought to be a boring playground for physics research. What is there to be found in a theory absent of degrees of freedom? This mentality has changed for good, as the discovery of different structures (black holes, asymptotics) and the inclusion of new ingredients (higher spin fields) has sparked novel research with ambitious goals, spanning from Low to High Energy Physics.

This thesis will hopefully have portrayed some of the enriched atmosphere alluded to above. We first discussed the special features of three dimensional gravity, with a specific focus in its gauge-theoretic description. This task requires translating common General Relativity tools into gauge theory ones, we review the most important ones for this project. Then, higher spin theory is motivated and introduced, and, in the context of three dimensions, extra novel features coming from these exotic theories are analyzed. Tying all the discussed context together, Wilson lines are used as topological probes of the spacetime, initially with the aim of finding an analog to the Ryu-Takayanagi proposal, but later studying them in more generality, including quantum corrections, finding their relation to propagators of minimally coupled scalar fields in locally AdS_3 backgrounds. We wrapped all up with the study of such Wilson lines in spin-3 gravity, whose quantum description was still lacking. There are many interesting phenomena from each topic which were left behind, the reader is strongly encouraged to navigate through the literature and learn about them.

The research undertaken in this project is far from complete. We have barely scratched the surface of the quantum mechanical description of the higher spin Wilson line, we just established the pillars and checked their sensibility. Some extra features were captured and generalized (not being restricted to a time slice, allowing for higher spin charged probes), though. Here I will recap some aspects which would be interesting to study in the future:

1. We have seen that the relation between α, β and c_2, c_3 involves $\Delta\alpha_2, \Delta\alpha_3$, which are spacetime-dependent. $\Delta\alpha_2$ can be though of a generalization of the geodesic distance, however $\Delta\alpha_3$ is more mysterious. In the diagonal embedding, where computations can be carried out without much difficulty, we can see that if the probe carries no higher spin $\Delta\alpha_3 = -4(\eta + 2q)\Delta\phi/3c_2$, from (6.58) and (6.67). One would need a similar computation for the principal embedding to see what behaviour of $\Delta\alpha_3$ is general in both cases.

2. In section 5.3 we discussed the role of the Wilson line as a propagator of a massive scalar coupled to gravity [4]. Said Wilson line was a solution to the Klein-Gordon equation for a

massive scalar in curved spacetime. Therefore, the Wilson line had the interpretation of a Green's function, hence a propagator. We hope that the higher spin Wilson line computed here is again a propagator, in a higher spin geometry.

Note however that here we are dealing with a considerably more complicated problem: first of all, the equation of motion should be cubic in derivatives. This is something very unusual, as equations of motion are generally either first or second order differential equations, never third. Secondly, a working metric formulation of these SL(N, $\mathbb{R}$) higher spin gravities is yet to be firmly established, despite attempts such as [41]. Not only that, but even a success in writing an appropriate action for this metric would not be enough, as we would need to couple a massive scalar to it, respecting the gauge symmetry. Until that task is completed, we do not know which equation would the higher spin Wilson line solve.

There is a simpler case that we can consider: as discussed above, the diagonal embedding can be consistently truncated to SL(2, $\mathbb{R}$)× U(1). The reason why it is simpler is that for this case there is a metric formulation. This would serve as a stepping stone for the full SL(3, $\mathbb{R}$) case, as it would involve a similar computation to that of SL(2, $\mathbb{R}$) plus an extra ingredient, the U(1) fields.

3. The Wilson line computed (6.41) was derived relying purely on group-theoretic results. However the motivation for this study began in [2] with the the semiclassical approach, which relied on the path integral formalism. In the SL(2, $\mathbb{R}$) case it was already shown that there was a connection between them, however a relation to this formulation is still lacking for the higher spin Wilson line. This computation would be interesting in order to confirm that the semiclassical result is recovered for $h, w \gg 1$.

4. At the level of evaluating the probe, there are many interesting features to explore. On the one hand, as we are assuming the Wilson line to be a propagator on these higher spin geometries, it is suitable for characterizing causality, as the metric tools are not reliable. This would give us a very good handle of these geometries. In order to do that, we need to find the singularities of the Wilson line (in the SL(2, $\mathbb{R}$) case, for instance, points separated by null geodesics have this singularity).

   Another evaluation test which would be useful is understanding the role of the radial gauge $b(\rho)$, how it affects the inner product.

5. In order to connect these results with four dimensional analogs it would be interesting to generalize them to SL(N, $\mathbb{R}$) or even to the higher spin algebras hs[λ], which are a step closer to Vasiliev's theory.

Appendix A

Vielbein Formalism

Any introductory course or lecture notes on General Relativity emphasizes the use of the well-known coordinate basis. However, we all know that physics should not depend on the basis vectors used to describe it, therefore if we pick, say, a combination of these basis vectors, we should end up finding the exact same results. But two questions can be raised upon considering this possibility: why would we do that and which combinations are interesting?

One of the first authors developing this formalism was Albert Einstein. Not directly because of his acclaimed 1915's General Theory of Relativity; but because of his quite unknown two letters attempting to unify gravity and electromagnetism. Unlike previous attempts to achieve this, Einstein tried not to include additional structure to General Relativity. As a counterexample, the famous Kaluza-Klein approach required one extra dimension and an extra scalar field, the dilaton. These letters were released in 1928 (soon after the discovery of the Dirac equation, being regarded as the "square root" of the Klein-Gordon equation; which could have inspired Einstein to write the above letters, as he introduces the concept of "square root" of the metric as well). Initially rejected by the physics community, the content of these letters has regained interest in the context of supergravity.

This formalism is completely necessary in order to regard General Relativity as a gauge theory, a task that has been successful in (2+1) dimensions [**?**, 12]. This appendix is intended as a comprehensive introduction to the main tools and results of this approach, and hopefully self-contained, as navigating through the literature on this formalism is usually a notational nightmare.

A.1 Non-coordinate basis

As introduced above, we are perfectly allowed to use any basis spanning the tangent space at a given point, $T_p\mathcal{M}$. A reasonable choice, besides the usual coordinate basis, is choosing vectors such that their inner product is given by $g(e_a, e_b) = \eta_{ab}$. The relevance of this choice will become clear as we continue.

The relation between both sets of vectors (bear in mind the chapter Notation and conventions; indices are the main reason why this topic is notationally harsh) is then

$$e_\mu(x) = e_\mu{}^a(x) e_a, \quad e_a = e^\mu{}_a(x) e_\mu(x); \quad e_\mu{}^a(x), e^\mu{}_a(x) \in \mathrm{GL}(n) \tag{A.1}$$

where $\mathrm{GL}(n)$ stands for the group of $n \times n$ invertible matrices, with $n = \dim \mathcal{M}$. The matrices $e_\mu{}^a$

are called[1] "vielbeins".

Orthonormality further implies

$$e^{\mu}{}_{a}(x)e_{\mu}{}^{b}(x) = \delta_{b}{}^{a}, \quad e^{\mu}{}_{a}(x)e_{\nu}{}^{a}(x) = \delta_{\nu}{}^{\mu} \tag{A.2}$$

If we now express the relation between the inner product of these basis vectors and the metric it is clear why the vielbeins are commonly called the "square root" of the metric:

$$g(e_a, e_b) = g_{\mu\nu}(x)e^{\mu}{}_{a}(x)e^{\nu}{}_{b}(x) = \eta_{ab} \tag{A.3}$$

Not only does this apply to vectors, it also works with covectors (1-forms). The orthonormality condition (A.2) allows us to write

$$\theta^{\mu} = e^{\mu}{}_{a}\theta^{a}, \quad \theta^{a} = e_{\mu}{}^{a}\theta^{\mu} \tag{A.4}$$

where we have already dropped the explicit dependence of the vielbeins in the coordinates, for the sake of simplicity.

In essence, the vierbeins can be used to change from one set of indices to the other on any tensor components, and the lowering-raising indices operations will need the appropriate metric

$$T^{\mu}{}_{\nu}{}^{a}{}_{b} = e^{\mu}{}_{c}\, e_{\nu}{}^{d}\, e_{\rho}{}^{a}\, e^{\sigma}{}_{b}\, T^{c}{}_{d}{}^{\rho}{}_{\sigma} = g^{\mu\rho}\, g_{\nu\sigma}\, \eta^{ac}\, \eta_{bd}\, T_{\rho}{}^{\sigma}{}_{c}{}^{d} \tag{A.5}$$

This is all one needs to know about changes between both sets of indices. But what about changes that preserve the inner product? We know from Special Relativity that the transformations that preserve the Minkowski metric are the ones belonging to the Lorentz group[2]:

$$\Lambda^{a}{}_{a'}\Lambda^{b}{}_{b'}\,\eta_{ab} = \eta_{a'b'}; \quad \Lambda^{a}{}_{a'} \in \mathrm{SO}(1,3) \tag{A.6}$$

Hence, transformations between different orthonormal basis are given by

$$e_{a'} = \Lambda^{a}{}_{a'}e_{a}, \quad e_{a} = \Lambda_{a}{}^{a'}e_{a'} \tag{A.7}$$

and generalizing for the case of the covectors is trivial. This result doesn't come as a surprise, in fact our intuition is correct in this case, as it tells us that orthonormal basis can be changed one into one another only by means of rotations. This also tells us something rather subtle about General Relativity: it has a hidden symmetry, namely local Lorentz invariance. We can therefore expect, and we will make it explicit very soon, that we need to associate a covariant derivative as is done in gauge theories.

For completeness, it should be clear how the tensor transformation law is modified for any kind of tensor, i.e.

$$T^{\mu'}{}_{\nu'}{}^{a'}{}_{b'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\,\Lambda_{a}{}^{a'}\,\Lambda^{b}{}_{b'}\,T^{\mu}{}_{\nu}{}^{a}{}_{b} \tag{A.8}$$

[1]One has to be careful when searching through the literature, as this term is used for both the set of basis vectors and the transformation matrices (it is also common to find them denoted as tetrad)

[2]Here we are treating the 4d Lorentzian case, but in the rest of the thesis we will be considering 3d gravity. The only change is the group, which will be SO(1,2). If we are instead considering an Euclidean geometry, the group becomes the usual special orthogonal in n-dimensions SO(n).

A.2 Differentiation

It can be asked whether there can be a notion of covariant derivative in the vierbein framework. Indeed, we will still need a correction term for a well-defined derivative; we are still working on curved space. The difference will be that the connection coefficients will no longer be the Christoffel symbols, but the spin connection[3] components, ${\omega_\mu}^a{}_b$, although just for the Latin indices, the Greek or curved ones will remain corrected by the usual Christoffel symbols. The covariant derivative then reads:

$$\nabla_\mu X^a{}_b = \partial_\mu X^a{}_b + {\omega_\mu}^a{}_c X^c{}_b - {\omega_\mu}^c{}_b X^a{}_c \tag{A.9}$$

However the spin connection and the Christoffel connection are not independent, and their relation can be made explicit by demanding independence of the basis of, for simplicity

$$\nabla X = (\nabla_\mu X^\nu)\mathrm{d}x^\mu \otimes \partial_\nu = (\nabla_\mu X^a)\mathrm{d}x^\mu \otimes e_a \tag{A.10}$$

This leads to

$$\Gamma^\nu{}_{\mu\lambda} = e^\nu{}_a {e_\lambda}^b {\omega_\mu}^a{}_b + e^\nu{}_a \partial_\mu {e_\lambda}^a, \quad {\omega_\mu}^a{}_b = {e_\nu}^a e^\lambda{}_b \Gamma^\nu{}_{\mu\lambda} - e^\lambda{}_b \partial_\mu {e_\lambda}^a \tag{A.11}$$

Another way to derive this is through the so-called "Tetrad postulate": imposing $\nabla_\mu {e_\nu}^a = 0$ implies the above relations.

An important fact to notice is that the Greek index of the spin connection does transform as a tensor, while the Latin ones transform inhomogeneously, in a similar fashion as the Christoffel symbols:

$${\omega_\mu}^{a'}{}_{b'} = {\Lambda_a}^{a'} \Lambda^b{}_{b'} {\omega_\mu}^a{}_b - \Lambda^c{}_{b'} \partial_\mu {\Lambda_c}^{a'} \tag{A.12}$$

We already have the ingredients necessary to understand why this formalism is so powerful: It allows us to think of some important tensors as differential forms, bringing all their machinery to life. For instance, a (1,1) mixed tensor ${X_\mu}^a$, can also be interpreted as a vector-valued one-form, $X^a = {X_\mu}^a\, \mathrm{d}x^\mu$, thus allowing us to exterior differentiate as

$$(\mathcal{D}X)_{\mu\nu}{}^a \equiv (\mathrm{d}X)_{\mu\nu}{}^a + (\omega \wedge X)_{\mu\nu}{}^a = \partial_\mu {X_\nu}^a - \partial_\nu {X_\mu}^a + {\omega_\mu}^a{}_b\, {X_\nu}^b - {\omega_\nu}^a{}_b\, {X_\mu}^b \tag{A.13}$$

where the $\omega \wedge X$ term has to be introduced in order to have the proper transformation law of the exterior derivative; in this sense we can think of $\mathcal{D}$ as an "exterior covariant derivative". Disguised as differential forms $e^a \equiv {e_\mu}^a\, \mathrm{d}x^\mu$, $\omega^a{}_b \equiv {\omega_\mu}^a{}_b\, \mathrm{d}x^\mu$, releasing some of the heavy notational weight. Using this, two definitions (also known as "Maurer-Cartan structure equations") arise in a fairly simple way,

$$T^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b; \quad \text{(Torsion 2-form)} \tag{A.14a}$$

$$R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b; \quad \text{(Curvature 2-form)} \tag{A.14b}$$

These 2-forms satisfy the Bianchi identities (well known from Differential Geometry or General Relativity):

$$\mathrm{d}T^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge e^b \longleftrightarrow R^\rho{}_{[\sigma\mu\nu]} = 0 \tag{A.15a}$$

$$\mathrm{d}R^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0 \longleftrightarrow \nabla_{[\lambda|} R^\rho{}_{\sigma|\mu\nu]} = 0 \tag{A.15b}$$

It can be shown that requiring the connection to be affine (therefore yielding the Christoffel symbols as its components) implies, in the language of the current discussion, that it is torsion-free, i.e. $T^a = 0$, and metric compatible, $\nabla g = \nabla \eta = 0 \Rightarrow \omega_{\mu ab} = -\omega_{\mu ba}$.

[3] One of the reasons why the spin connection is so relevant is that it allows, as its name foretells, the introduction of spinors in curved spacetime, something which is not possible with the usual connection.

Appendix B

SL(2, ℝ) and SL(3, ℝ) conventions

The special linear group SL(N, ℝ) with $N \in \mathbb{Z}^+$ is a Lie group whose elements are $N \times N$ matrices with determinant 1 and whose group operation is the usual matricial product. We will also be interested in their corresponding Lie algebras $\mathfrak{sl}(N, \mathbb{R})$. A convenient basis for these Lie algebras is the one making explicit the $\mathfrak{sl}(2, \mathbb{R})$ subalgebra, represented by $\{L_0, L_{\pm 1}\}$, while the rest of the generators are denoted $W_j^{(s)}$ with this choice, their commutation relations are

$$[L_m, L_n] \ = \ (m-n)\, L_{m+n}, \tag{B.1a}$$

$$[L_n, W_j^{(s)}] \ = \ [n(s-1)-j]\, W_{n+j}^{(s)}, \quad |j| < s-1. \tag{B.1b}$$

Particularly relevant algebras in this thesis will be those of the $N = 2$ and $N = 3$ cases. Conventions used for representations of these algebras can be found below.

B.1 Representations of $\mathfrak{sl}(2, \mathbb{R})$

The $\mathfrak{sl}(2, \mathbb{R})$ algebra is

$$[L_i, L_j] \ = \ (i-j)\, L_{i+j} \tag{B.2}$$

Our convention for the fundamental representation of $\mathfrak{sl}(2, \mathbb{R})$ will be

$$L_1 \ = \ \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad L_{-1} \ = \ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad L_0 \ = \ \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.3}$$

With this choice, the Killing form becomes

$$\eta_{ab} = \mathrm{Tr}_f(L_a L_b). \tag{B.4}$$

B.2 Representations of $\mathfrak{sl}(3, \mathbb{R})$

The $\mathfrak{sl}(3, \mathbb{R})$ is

$$[L_i, L_j] \ = \ (i-j)\, L_{i+j}\,, \tag{B.5a}$$

$$[L_i, W_m] \ = \ (2\,i-m)\, W_{i+m}\,, \tag{B.5b}$$

$$[W_m, W_n] \ = \ -\frac{1}{3}\,(m-n)\,(\,2\,m^2+2\,n^2-mn-8\,)\, L_{m+n}\,, \tag{B.5c}$$

For the fundamental representation of $\mathfrak{sl}(3,\mathbb{R})$ we will use

$$
\begin{aligned}
L_0 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, & L_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & L_{-1} &= \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \\
W_0 &= \frac{2}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & W_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, & W_{-1} &= \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \\
W_2 &= 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & W_{-2} &= 2\begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
\tag{B.6}
$$

In this case the quadratic and cubic Killing forms will be, respectively

$$
\delta_{ab} = \mathrm{Tr}_f(T_a T_b), \quad h_{abc} = \mathrm{Tr}_f(T_{(a} T_b T_{c)}) \tag{B.7}
$$

where $T_a = \{L_i, W_n\}$.

B.2.1 Conformal weight and $\mathfrak{sl}(2,\mathbb{R})$ spin:

Here we introduce and clarify frequently used terminology in the higher spin literature: a field $\phi(z)$ has conformal weight J if under $z \to z'(z)$ it transforms as

$$
\phi'(z') = \left(\frac{\mathrm{d}z}{\mathrm{d}z'}\right)^J \phi(z). \tag{B.8}
$$

If we expand in modes and given that $\mathfrak{sl}(2,\mathbb{R})$ generates the infinitesimal transformation, we find

$$
[L_m, \phi_n] = [m(J-1) - n]\phi_{m+n}, \tag{B.9}
$$

thus the W_n modes have conformal weight 3 while the L_n's have conformal weight 2. This bracket can be modified to contain a central extension.

This above definitions are related but distinct to the $\mathfrak{sl}(2,\mathbb{R})$ spin [23]. Given J_a basis of $\mathfrak{sl}(2,\mathbb{R}) \in \mathfrak{sl}(N,\mathbb{R})$, the rest of the $\mathfrak{sl}(N,\mathbb{R})$ generators will form a representation of $\mathfrak{sl}(2,\mathbb{R})$, labelled by their spin. The dimension D of this representations are related to its spin s as $D = 2s+1$, that is why in the principal embedding of $\mathfrak{sl}(3,\mathbb{R})$ the $\{W_{\pm 2}, W_{\pm 1}, W_0\}$ transform in the spin-2 representation of $\mathfrak{sl}(2,\mathbb{R})$ and analogously for the spin-0 and $2\times$ spin-1/2 of the diagonal embedding.

Appendix C

Entanglement entropy and the Ryu-Takayanagi proposal

The phenomenon of entanglement poses challenges when it comes to understand physical systems, due to the information loss when accessing one of the entangled subsystems. Entanglement entropy is a useful measure of the information available from the system, and in the last decades it has been an active area of research, from the understanding of black holes and the Information Paradox to the surge of Quantum Information and Quantum Computing.

In this appendix entanglement entropy and related concepts, which will play a role mainly in Chapter 5, will be briefly reviewed. At the end of this appendix an overview of the Ryu-Takayanagi proposal will be given, which threads all these concepts into the bulk of the project. For more in-depth reviews of all the material covered in this appendix we refer to [42, 43].

C.1 Entanglement entropy in Quantum Mechanics and Quantum Field Theory

Given two Hilbert spaces H_A, H_B, the most general state in their composite system $H_A \otimes H_B$ is:

$$|\psi\rangle = \sum_{i,j} c_{ij} |A_i\rangle \otimes |B_j\rangle \,. \tag{C.1}$$

This state will be entangled in general, as it cannot be separable into both subsystems, i.e. we generally have $c_{ij} \neq c_i^A c_j^B$. The entropy is a magnitude related to the amount of information accessible from a system, and therefore it's perfectly adequate to study entangled states. In order to characterize it, we resort to the density matrix of the system,

$$\rho = |\psi\rangle\langle\psi| \,, \tag{C.2}$$

The density matrix is something not only available for pure states such as (C.1), but also for mixed states without a pure state description. Some of the properties of the density matrix are hermiticity $\rho^\dagger = \rho$, positivity of its eigenvalues $\rho \geq 0$ and unit trace $\mathrm{Tr}\rho = 1$.

If we want to access the information available in subsystem A, we are blind to its complementary subsystem, B. Mathematically this means that we work with the reduced density matrix ρ_A, obtained by tracing out subsystem B,

$$\rho_A = \mathrm{Tr}_B\,\rho = \sum_i \langle B_i|\rho|B_i\rangle\,. \tag{C.3}$$

This is a measure of the degrees of freedom accessible from subsystem A. The entanglement entropy is then given by the von Neumann entropy,

$$S_{\mathrm{EE}} = -\mathrm{Tr}_A\,(\rho_A \log \rho_A). \tag{C.4}$$

This is the quantum version of the Shannon entropy, and is related to the entropy of configurations with a certain number of particles in a certain number of states, at the same energy.

Notice that the operation of taking the logarithm of a matrix does not seem to be well defined. In general it refers to taking the logarithm of its eigenvalues, however there is a way to circumvent this obstacle which goes by the name of the replica trick:

$$S_{\mathrm{EE}} = -\frac{\partial}{\partial n}\mathrm{Tr}_A\,\rho_A^n\bigg|_{n\to 1} = -\mathrm{Tr}_A\,(\rho_A^n \log \rho_A)\bigg|_{n\to 1} = -\mathrm{Tr}_A\,(\rho_A \log \rho_A), \tag{C.5}$$

where in order to take the derivative we have analytically continued n to the real line. Alternatively, one can compute the n-th Renyi entropy

$$S^{(n)} = \frac{1}{1-n}\log \mathrm{Tr}_A\,\rho_A^n, \tag{C.6}$$

with the entanglement entropy as the limit $S_{\mathrm{EE}} = \lim_{n\to 1} S^{(n)}$. This trick removes the trouble coming from the logarithm of a matrix, which is exchanged for a "simple" operation, tracing powers of a matrix.

All the above has already been generalized to Quantum Field Theories and Conformal Field Theories [33], which involves some technology such as an Euclidean path integral treatment of states, replica fields, etc. For 2d CFT's some important results have been computed and will be referred to in Chapter 5, which we list now

- CFT in the vacuum, single interval

$$S = \frac{c}{3}\log\left(\frac{L}{\epsilon}\right). \tag{C.7}$$

- CFT in a thermal state, single interval

$$S = \frac{c}{3}\log\left(\frac{1}{\pi\beta\epsilon}\sinh\frac{\pi L}{\beta}\right). \tag{C.8}$$

where in both cases L is the length of the interval, c is the central charge and ϵ is the UV cutoff.

C.2 Ryu-Takayanagi proposal

In 2006, Ryu and Takayanagi showed [1] that the entanglement entropy of a subregion in CFT_d takes the form

$$S_{\mathrm{EE}} = \frac{\mathrm{Area}(\gamma)}{4G_N^{d+1}}, \tag{C.9}$$

where γ is a minimal surface in the AdS bulk with its boundary $\partial\gamma$ being the entangling surface of the CFT. It is also clear from this that if we consider a minimal surface wrapping a black hole in AdS, it should reduce to its Bekenstein-Hawking entropy.

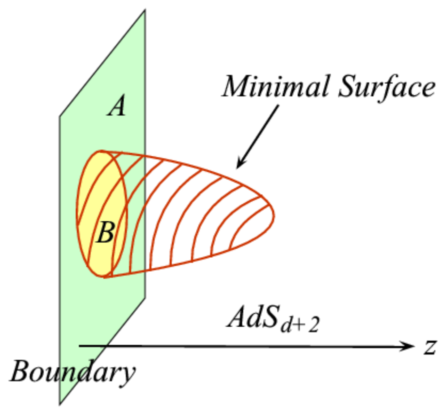

Figure C.1: Visualization of the Ryu-Takayanagi proposal. Figure from [42].

This "Area law" proposal is backed up with several computations [1, 42, 43] and passes tests such as subadditivity and strong subadditivity (which are properties known to characterize entanglement entropies).

Appendix D

Generalization to $\mathrm{SL}(4,\mathbb{R})$

In order to come up with a general formula, we will study the case $\mathfrak{sl}(4,\mathbb{R})$ making use of what we discovered in the usual gravity and spin-3 case. We will take as granted some features of these simpler cases (although a thorough computation should be carried out to check the final result). Namely,

1. The highest weight representation will be assumed to be unitary as in the $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{sl}(3,\mathbb{R})$ cases, there is no reason to think that non-algebra null states will appear in this one.

2. The form of the Ishibashi/crosscap states will also be chosen to be the simplest one: a sum over diagonal (e.g. $\bar{k} = k$) values of the descendants' labels associated to the even weight operators, $W^{(2)}, W^{(4)}, \ldots$, a sum over antisymmetric $\bar{s} = -s$ for the odd ones $W^{(3)}, \ldots$ and coefficients ± 1 before each of them. The actual sign, + or − depends on the state being Ishibashi or crosscap and will not play a role when computing the inner product as, $(\pm)^2 = +$.

With this in mind, let us then follow the same steps as in the $\mathfrak{sl}(3,\mathbb{R})$ case.

D.1 Diagonalization, and highest weight representation

Let's begin building the highest weight representation: a highest weight state $|h,w,u\rangle$ is postulated to exist, satisfying

$$\begin{aligned} L_0\,|h,w,u\rangle = h\,|h,w,u\rangle\,, \quad W_0\,|h,w,u\rangle = w\,|h,w,u\rangle\,, \quad U_0\,|h,w,u\rangle = u\,|h,w,u\rangle \\ L_n\,|h,w,u\rangle = W_n\,|h,w,u\rangle = U_n\,|h,w,u\rangle = 0,\ n>0 \end{aligned} \tag{D.1}$$

In the fundamental representation of $\mathfrak{sl}(4,\mathbb{R})$ (for computational ease we will initially work out the diagonal basis in this representation), we have the following 15 generators (see [44], however I picked a different normalization of the U's):

$$L_0 = \frac{1}{2}\begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \qquad L_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix},$$

$$W_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad W_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad W_{-1} = 3\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$W_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad W_{-2} = 12\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad U_0 = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$U_1 = \frac{1}{5}\begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad U_2 = \frac{5}{6}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad U_3 = \frac{5}{3}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$U_{-1} = 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad U_{-2} = 10\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad U_{-3} = -60\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{D.2}$$

The full algebra of the generators will be of no interest to us, we only need to know their relation to the Cartan subalgebra, spanned by L_0, W_0, U_0:

$$[L_0, W_0] = [L_0, U_0] = [W_0, U_0] = 0. \tag{D.3}$$

We have to find the basis diagonalizing simultaneously the three Cartans. It will consist of 12 operators, out of which the 6 raising ones are obtained as

$$M^{(\pm)} = L_{-1} \pm \frac{5}{3} W_{-1} + U_{-1}, \quad M^{(0)} = L_{-1} - \frac{3}{2} U_{-1}, \quad D^{(\pm)} = W_{-2} \pm \frac{6}{5} U_{-2}, \quad U_{-3}. \tag{D.4}$$

In this basis, we find the weights of the operators to be (remember that they are all lowering, albeit the ± indices)

$$\begin{aligned} [L_0, M^{(\pm)}] = M^{(\pm)}, \quad [L_0, M^{(0)}] = M^{(0)}, \quad [L_0, D^{(\pm)}] = 2\,D^{(\pm)}, \quad [L_0, U_{-3}] = 3\,U_{-3}, \\ [W_0, M^{(\pm)}] = \pm 2\,M^{(\pm)}, \quad [W_0, M^{(0)}] = 0, \quad [W_0, D^{(\pm)}] = \pm 2\,D^{(\pm)}, \quad [W_0, U_{-3}] = 0, \\ [U_0, M^{(\pm)}] = 2\,M^{(\pm)}, \quad [U_0, M^{(0)}] = -3\,M^{(0)}, \quad [U_0, D^{(\pm)}] = -D^{(\pm)}, \quad [U_0, U_{-3}] = U_{-3}. \end{aligned} \tag{D.5}$$

We can now find the tower of states generated by these raising operators acting on the highest weight state, which we shorten as $|h, w, u; k, s, z\rangle \equiv |k, s, z\rangle$.

$$\begin{aligned} M^{(\pm)}\,|k, s, z\rangle \propto |k+1, s \pm 2, z+2\rangle\,, \quad M^{(0)}\,|k, s, z\rangle \propto |k+1, s, z-3\rangle\,, \\ D^{(\pm)}\,|k, s, z\rangle \propto |k+2, s \pm 2, z-1\rangle\,, \quad U_{-3}\,|k, s, z\rangle \propto |k+3, s, z+1\rangle\,, \end{aligned} \tag{D.6}$$

As in the case of $\mathfrak{sl}(3,\mathbb{R})$, the level 1 generators $M^{(+,0,-)}$ completely determine the tower of states (here is where the first bullet point stated before comes into play, as we will assume that there are no null states because of this redundancy between generators). Hence, we find Figure D.1.

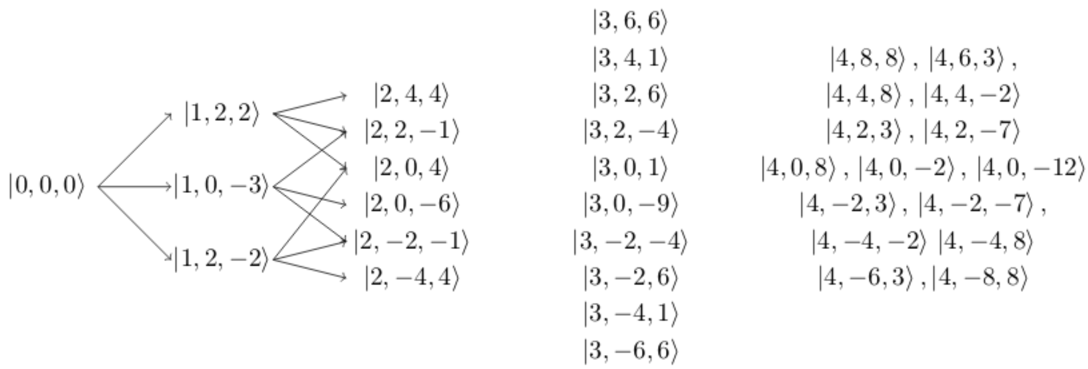

Figure D.1: Highest weight representation of $\mathfrak{sl}(4,\mathbb{R})$. Here it is depicted in a similar fashion as the $\mathfrak{sl}(3,\mathbb{R})$ one, however the correct way would be to represent it in three dimensions, one for each of the sublabels k, s, z.

D.2 Tower of states & Ishibashi/crosscap

The hard task is now coming up with a formula that captures this whole tower of states, without double-counting. Putting all our pattern-recognition skills at work, we "peel" the tower layer by layer:

1. To begin, notice that the $\mathfrak{sl}(3,\mathbb{R})$ tower of states appears, having as spin-4 label twice the spin-2 one. Therefore, we peel them off the full tower, as we can just generate them using the $\mathfrak{sl}(3,\mathbb{R})$ formula:

$$\begin{aligned} |\text{Sum}_1\rangle &= \sum_{k=0}^{\infty}\left(\sum_{s=-k}^{k} d_{2k,4s,4k}\,|2k,4s,4k\rangle + \sum_{s=-k}^{k+1} d_{2k+1,4s-2,4k+2}\,|2k+1,4s-2,4k+2\rangle\right) \\ &= |0,0,0\rangle + |1,2,2\rangle + |1,-2,2\rangle + |2,4,4\rangle + |2,0,4\rangle + |2,-4,4\rangle \dots \end{aligned} \tag{D.7}$$

2. Next, at first level we are missing the state generated by $M^{(0)}$. This turns out to happen at every level above the zeroth one, hence we summon them as

$$|\text{Sum}_2\rangle = \sum_{k=1}^{\infty} d_{k,0,-3k}\,|k,0,-3k\rangle = |1,0,-3\rangle + |2,0,-6\rangle + \dots \tag{D.8}$$

   so we can remove them from the tower.

3. The above is accurate up to first level, but at second and above there are more states. They are symmetrical with respect to $s = 2$, so we are going to find two families, one with $s > 0$

and the same one with $s < 0$. The way to generate them is

$$
\begin{aligned}
|\text{Sum}_3\rangle &= \sum_{k=2}^{\infty}\sum_{z=2}^{k}\left(d_{k,2z-2,5z-3k-5}\,|k,2z-2,5z-3k-5\rangle + d_{k,-2z+2,5z-3k-5}\,|k,-2z+2,5z-3k-5\rangle\right) \\
&= |2,2,-1\rangle + |2,-2,-1\rangle + \dots
\end{aligned}
\tag{D.9}
$$

4. Sadly, this is not the end of the story. At third level an unaccounted state appears, at fourth level 3 appear, etc. However, we have seen that pattern before: it is this exact $\mathfrak{sl}(4,\mathbb{R})$ tower once again, but starting with $|3k,0,k\rangle$ instead of the highest weight. Hence, if we want to capture all the states of the tower, we need to add an extra index to each and every state we already found, to replicate them, e.g.

$$
|k,s,z\rangle \to \sum_{n=0}^{\infty} d_{k+3n,s,z+n}\,|k+3n,s,z+n\rangle \tag{D.10}
$$

Considering all these sums, we find exact agreement with the tower of states, which has been checked up to 8th level, and there is no reason to think that it will not hold for the rest of them. Note that the claim is not that this is the unique way to generate the tower; there might be a different approach, with different (more clever) sums such that it also describes it. However, as this one works, we will continue with it. For visual aid, Figure D.2 depicts the above bullet points.

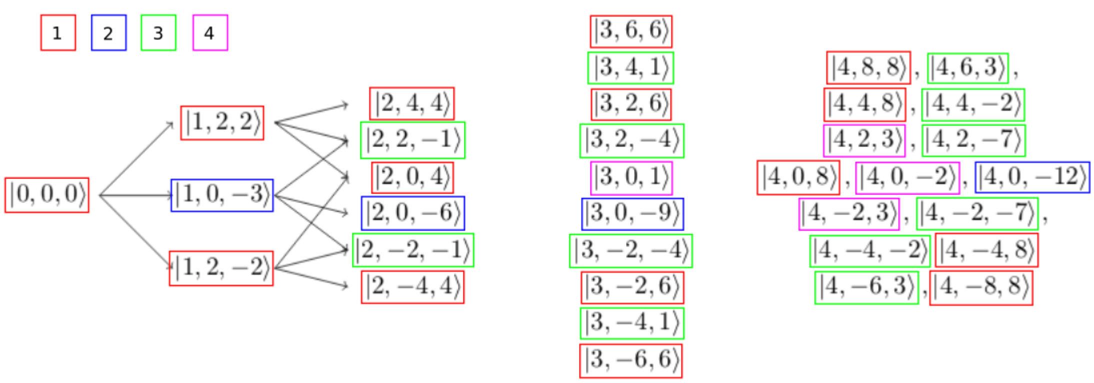

Figure D.2: Visualization of the four bullet points above and the states which they generate. The color code is included in the picture.

The good news is that when computing the inner product, as the group element will be diagonal, each state overlaps only with itself. The Ishibashi state (assuming the above second bullet point), is

$$
\begin{aligned}
|\Sigma\rangle &= \sum_{n=0}^{\infty} \Big[\, |\mathrm{Sum}_1\rangle \otimes \overline{|\mathrm{Sum}_1\rangle} + |\mathrm{Sum}_2\rangle \otimes \overline{|\mathrm{Sum}_2\rangle} + |\mathrm{Sum}_2\rangle \otimes \overline{|\mathrm{Sum}_3\rangle} \Big] \\
&= \sum_{n=0}^{\infty} \Bigg[\sum_{k=0}^{\infty} \Bigg(\sum_{s=-k}^{k} |h,w,u;2k+3n,4s,4k+n\rangle \otimes |h,-w,u;2k+3n,-4s,4k+n\rangle \\
&\quad + \sum_{s=-k}^{k+1} |h,w,u;2k+1+3n,4s-2,4k+2+n\rangle \otimes |h,-w,u;2k+1+3n,-4s+2,4k+2+n\rangle \Bigg) \\
&\quad + \sum_{k=1}^{\infty} |h,w,u;k+3n,0,-3k+n\rangle \otimes |h,-w,u;k+3n,0,-3k+n\rangle \\
&\quad + \sum_{k=2}^{\infty} \sum_{z=2}^{k} \big(|h,w,u;k+3n,2z-2,5z-3k-5+n\rangle \otimes |h,-w,u;k+3n,-2z+2,5z-3k-5+n\rangle \\
&\quad + |h,w,u;k+3n,-2z+2,5z-3k-5+n\rangle \otimes |h,w,u;k+3n,2z-2,5z-3k-5+n\rangle \big) \Bigg],
\end{aligned}
\tag{D.11}
$$

with the crosscap state being this one up to - signs, which will be cancelled upon computing the inner product. Something that should be noted is that this formula is okay for the infinite tower, however if one wants to truncate it to all states up to certain level, the range of the n sum has to be modified. We will not worry about that here.

D.3 Inner product and analysis

The group element we are interested in is the following:

$$
G = \exp(-\alpha L_0 - \beta W_0 - \gamma U_0). \tag{D.12}
$$

Computing the braket of this element with the Ishibashi state above (D.11), and taking into account the degeneracies, we find

$$
\langle\Sigma|G|\Sigma\rangle = \frac{e^{-\alpha h-\beta w-\gamma u}}{\left(e^{3\gamma}-e^{-\alpha}\right)\left(e^{-2\beta-2\gamma}-e^{-\alpha}\right)\left(e^{2\beta-2\gamma}-e^{-\alpha}\right)\left(e^{-2\beta+\gamma}-e^{-2\alpha}\right)\left(e^{2\beta+\gamma}-e^{-2\alpha}\right)\left(e^{-\gamma}-e^{-3\alpha}\right)} \tag{D.13}
$$

This is the $\mathfrak{sl}(4,\mathbb{R})$ character.

Limits: If we take the $\gamma \to 0$ limit we find

$$
\lim_{\gamma\to 0} \langle\Sigma|G|\Sigma\rangle = \frac{e^{-\alpha h-\beta w}}{\left(1-e^{-\alpha}\right)\left(e^{-2\beta}-e^{-\alpha}\right)\left(e^{2\beta}-e^{-\alpha}\right)\left(e^{-2\beta}-e^{-2\alpha}\right)\left(e^{2\beta}-e^{-2\alpha}\right)\left(1-e^{-3\alpha}\right)} \tag{D.14}
$$

and if we further take the $\beta \to 0$ limit,

$$
\lim_{\beta,\gamma\to 0} \langle\Sigma|G|\Sigma\rangle = \frac{e^{-\alpha h}}{(1-e^{-\alpha})^6}. \tag{D.15}
$$

The pattern is clear: for the pure gravity case of $\mathfrak{sl}(N,\mathbb{R})$, we will have

$$\lim_{\text{pure grav.}} \langle\Sigma|G|\Sigma\rangle = \frac{e^{-\alpha h}}{(1-e^{-\alpha})^{\frac{N^2-N}{2}}}. \tag{D.16}$$

The exponent $\frac{N^2-N}{2}$ indicates the number of raising operators of $\mathrm{SL}(N,\mathbb{R})$. As discussed in subsection 6.3.2, this is not surprising in comparison with the usual CFT characters of the Virasoro algebra.

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