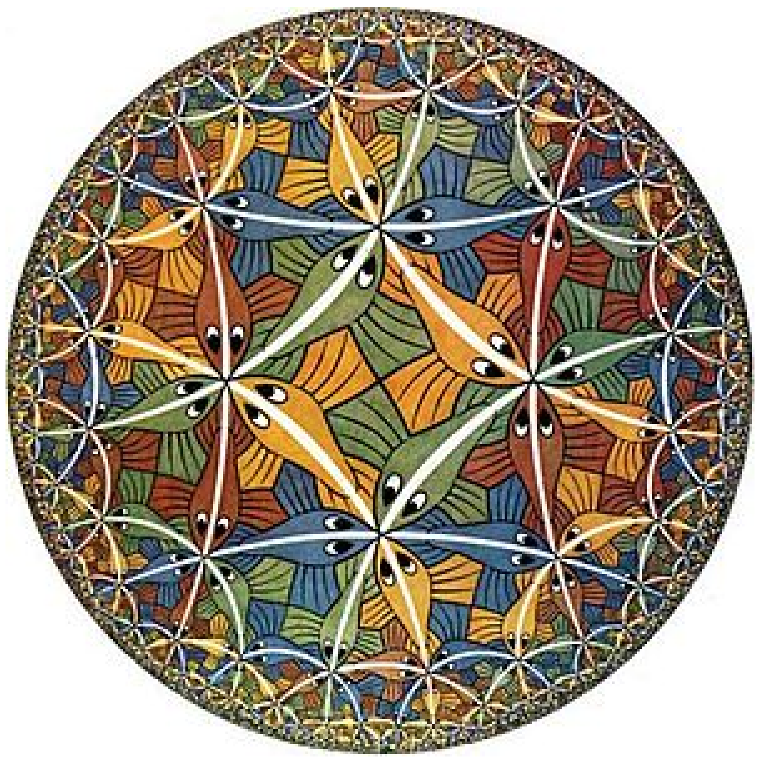
String Theory; extension

An introduction to the AdS/CFT conjecture



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Notation and conventions

Units: Throughout this lecture notes we will (usually) be using natural units, where all constants of nature are normalized to 1:

$$c = G_N = k_B = \hbar = 1$$

This makes equations much simpler, and the results are the same once we recover the dimensionality with the appropriate combination of these constants.

Dimensions: The correspondence states that a conformal field theory in d dimensions is dual to a gravitational theory in $d+1$. We will stick to this convention for each theories dimension.

Metric signature: The metric will have the "mostly plus" signature convention:

$$\mathrm{sign}(g) = (-, +, +, +, ...)$$

This is the standard notation in General Relativity references (with few exceptions), and it is the exact opposite to most of the particle physics literature. Beware.

Sometimes we will switch to Euclidean signature by means of a Wick rotation. This will be clear as we will use t for Lorentzian time, and t_E for Euclidean, with $t = it_E$

Indices: Spacetime indices will be denoted by Greek letters $\mu, \nu, ...$ while just spatial ones will be denoted by Latin letters $i, j, ...$

Einstein sum: Einstein summation convention (same indices up and down are implicitly summed over) will be used again and again:

$$\sum_a v_a w^a \equiv v_a w^a$$

The range of the index will be clear from the context, otherwise we will specify it.

Preamble

The goal of this course is to explain the foundations of the recently discovered AdS/CFT duality or correspondence, which has become a hot topic since its discovery in 1997 by Juan Maldacena.

But what is a duality? In the context of physics, two different theories are said to be dual when they have the same behaviour, same predictions, etc. Therefore one can use either of them to perform a calculation, whose result is independent of the choice made. Examples of this are the electric-magnetic duality (in the absence of sources), or the string dualities (S-duality, T-duality). The duality we will be interested in during this course relates two very different theories, a bulk gravitational one (more specifically, an Anti-de Sitter one), and a Quantum Field Theory (more specifically, a Conformal Field Theory one) which lives in the boundary of the Anti-de Sitter spacetime. At the level of the partition functions,

$$\mathcal{Z}_{\text{AdS}} = \mathcal{Z}_{\text{CFT}} \tag{0.1}$$

with a rather interesting (and useful) feature, namely that when one side of the duality is weakly coupled, then the other side is strongly coupled.

The original 1997 article (which featured the first known example of the duality, $\text{AdS}_5/\text{CFT}_4$), *The Large N Limit of Superconformal Field Theories and Supergravity* [1], is the most cited paper in the history of High Energy Physics, having accumulated more than 14000 (the number keeps increasing) citations; and subsequent papers on the topic by Witten [2] and Gubser, Klebanov, Polyakov [3] are also following its lead.

The reason behind this success is the vast range of applicability and implications of the correspondence, reaching fields from different areas such as Quantum Gravity, Quantum information, Condensed Matter Physics, Number Theory,...

Although the course intends to be self-contained, for the interested student we refer you to the book *Introduction to the AdS/CFT Correspondence* by Horațiu Năstase [4], and specifically for the chapter 2 (supersymmetry) part we recommend Peter West's *Introduction To Supersymmetry And Supergravity* [5], and Matteo Bertolini's *Lectures on Supersymmetry* [6]. At the end of the lecture notes the reader can also find a list of the main references used throughout the text.

The notes are organized as follows: lectures 1 and 2 will focus on the bulk side of the correspondence, AdS. From the geometry of AdS itself to scalar fields propagating in it, everything covered will be very general (any dimension). Lectures 3 and 4 will cover Supersymmetry, first an introduction to the formalism and then some explicit realizations. In the end we will introduce 't Hooft's limit, though not being restricted to SUSY, is of great importance there. Lecture 5 will be devoted to the actual derivation of the correspondence, in a similar way as Maldacena in his famous paper. The remaining lectures will cover several applications and features of the correspondence, from explicit tests (lecture 6) to renormalization from the point of view of holography (lecture 7), and other examples of the correspondence (lecture 8).

Note to the reader: this notes are not a substitution of the lectures and in fact some sections are not yet finished. Not only that, they will surely contain some mistakes/typos. If you spot them, please don't hesitate contacting Gonzalo, Alex or Marcos

Lecture 1

The geometry of Anti-de Sitter space

Our first step towards understanding the correspondence is General Relativity. This theory of gravity will define the bulk side of the correspondence. Concretely, we need to understand one solution to the Einstein's equations (the core of General Relativity):

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu}; \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{1.1}$$

The solution of interest in our case (which is a vaccum solution, i.e. $T_{\mu\nu} = 0$) is called Anti-de Sitter spacetime, and we will review its most remarkable properties extensively throughout this first chapter.

A sufficient amount of expertise in basic differential geometry (tensor manipulation, computation of curvature tensors, etc) will be assumed.

1.1 Anti-de Sitter as an embedding space, and coordinate patches

We all know from General Relativity that the Einstein field equations can be derived from a variational principle: one can define an action from the gravitational field whose Euler-Lagrange equations under the extremization of the metric tensor are this equations (1.1). It can be shown that the correct action is the so-called Einstein-Hilbert one

$$S_{\text{EH}} = \frac{1}{16\pi G} \int d^{d+1}x \sqrt{-g}(R - 2\Lambda) \tag{1.2}$$

We will need this lagrangian formalism later on in the course.

Anti-de Sitter (AdS) space is a solution to the Einstein's equations in the vacuum with a negative cosmological constant ($\Lambda < 0$). For a general dimension $d + 1$, it is convenient to express this constant in terms of what is called "the AdS radius", ℓ:

$$\Lambda = -\frac{d(d-1)}{2\ell^2} \tag{1.3}$$

But what do we mean when we say AdS "radius"? To make sense of this it is very useful to consider the following setting: the starting point is a space $\mathbb{R}^{2,d}$ (with this we basically mean flat space with a signature including two minus signs instead of just one), whose metric is

$$ds^2 = -(dX^{-1})^2 - (dX^0)^2 + \sum_{i=1}^{d} (dX^i)^2 \tag{1.4}$$

AdS space is understood as a hypersurface embedded in this space, whose equations is

$$-(X^{-1})^2 - (X^0)^2 + \sum_{i=1}^{d}(X^i)^2 = -\ell^2 \tag{1.5}$$

This explains why ℓ is nothing more mysterious than a radius.

Example (2-sphere embedded in euclidean 3d space): A very good analogy is the following: Consider $\mathbb{R}^3$, with the standard metric

$$ds^2 = dx^2 + dy^2 + dz^2$$

A 2-sphere can be understood as the hypersurface

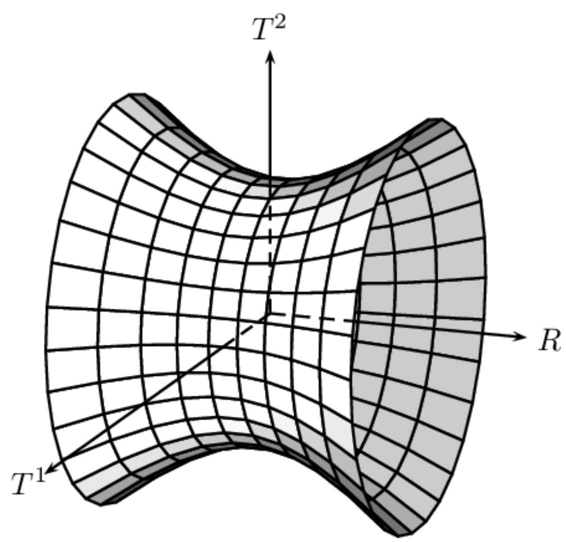
$$x^2 + y^2 + z^2 = R^2$$

which is embedded in the full 3d space. Here it is trivial to see why R is the radius.

The simplest version of AdS, still representing well its features is AdS_2, again parametrized by the hypersurface

$$(X^{-1})^2 + (X^0)^2 - (X^1)^2 = \ell^2$$

This can be drawn as



(where the notation of the picture is $R \equiv X^1$, $T^1 \equiv X^{-1}$, $T^2 \equiv X^0$)
We can see clearly from this image that as well as with the cylinder, it can be very useful if we parametrize the hypersurface with angles. Those will be called Global Coordinates of AdS:

$$\begin{cases} X^{-1} &= \ell \cosh\rho \sin t \\ X^0 &= \ell \cosh\rho \cos t \\ X^i &= \ell \sinh\rho \, \Omega^i \end{cases} \tag{1.6}$$

with Ω^i being the $(d-1)$ dimensional sphere.

Although it helps visualizing the situation, and it can be very useful in some scenarios, we are not interested in the whole "embedded-embedding" picture: we want to study spacetimes by their own glory. For example, when we say that the Universe experienced a de Sitter phase (the same as Anti-de Sitter but with $\Lambda > 0$) during inflation, we don't mean that the Universe was embedded in something else; it was a de Sitter spacetime by its own right.

Therefore, we need to obtain the metric of an embedded hypersurface, which is given by:

$$g^{\text{hyp}}_{\alpha\beta} = \frac{\partial X^\mu}{\partial x^\alpha} \frac{\partial X^\nu}{\partial x^\beta} g_{\mu\nu} \tag{1.7}$$

In our case $x^\alpha = (t, \rho, \Omega^i)$, hence we find

$$ds^2_{\text{AdS}} = \ell^2(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega^2_{d-1}) \tag{1.8}$$

This brings a problem that we could have seen from the previous picture: In the hyperboloid there are closed timelike curves. Causality is not satisfied. In order to fix this we have to "unwrap time"

$$S^1 \longrightarrow \mathbb{R} \tag{1.9}$$

$$0 < t_{\text{old}} < 2\pi \Longleftrightarrow t_{\text{old}} \sim t_{\text{old}} + 2\pi \longrightarrow \text{Universal cover of AdS} : -\infty < t < \infty \tag{1.10}$$

The result is a coordinate patch called Global AdS.

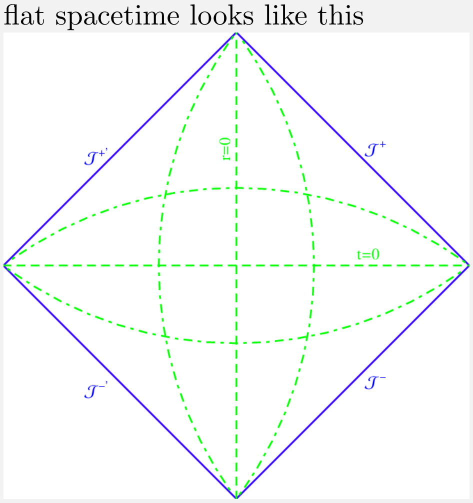
A slightly different patch, which we will use later and is quite common in the literature, can be obtained with the transformation $r = \ell \sinh \rho; \quad t \to t/\ell$:

$$ds^2_{\text{glob}} = -\left(\frac{r^2}{\ell^2} + 1\right) dt^2 + \left(\frac{r^2}{\ell^2} + 1\right)^{-1} dr^2 + r^2\, d\Omega^2_{d-1} \tag{1.11}$$

1.1.1 Causal structure and Penrose diagrams

Penrose diagrams are conformal compactifications of spacetime, where the causal structure can be made explicit as in them light rays are always depicted as travelling at 45^o degrees, just like in the spacetime we all know and love, Minkowski space.

Example (Penrose diagram of Minkowski space): The conformal compactification of flat spacetime looks like this



However, it can be seen from this picture that the conformal boundary of this spacetime is at 45^o. This is a null boundary, and is not good for holography: Quantum Field Theory is not well behaved there. As we will see later, AdS doesn't suffer this problem, its boundary behaves correctly.

We are going to change coordinates once more, to bring the coordinates to a bounded range (conformal compactification) $\tan\theta = \rho \to 0 \leq \rho < \infty \Rightarrow 0 \leq \theta \leq \pi/2$

$$ds^2 = \frac{\ell^2}{\cos^2\theta}(-dt^2 + d\theta^2 + \sin^2\theta d\Omega^2_{d-1}) \tag{1.12}$$

This is nice, because up to conformal transformations, we can strip off the front factor (it doesn't affect the causal structure).

We are left then with the metric of a sphere, although only covering the upper half, due to the range of θ.

$$ds^2 \to -dt^2 + d\theta^2 + \sin^2\theta d\Omega_{d-1}^2, \quad 0 \leq \theta \leq \pi/2 \tag{1.13}$$

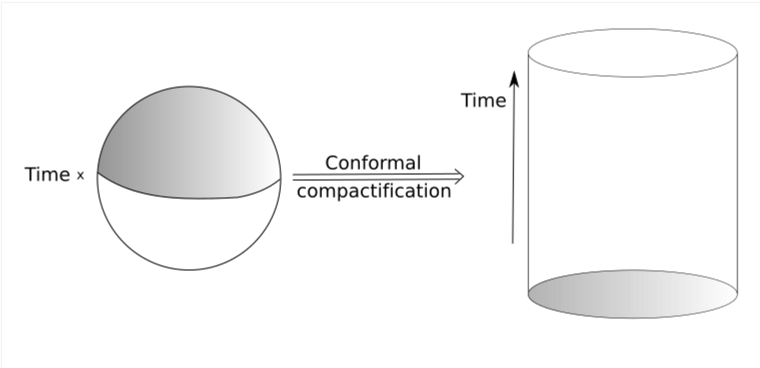


Figure 1.1: Penrose diagram for AdS

Now we have a timelike boundary! If we go to there ($\theta = \pi/2$),

$$ds_{\text{bdy}}^2 = -dt^2 + d\Omega_{d-1}^2; \quad \mathbb{R} \times S^{d-1} \xrightarrow[\text{conformally like}]{} \mathbb{R}^{1,d-1} \tag{1.14}$$

Can we find another patch such that the boundary is Minkowski? The answer is Yes. Let us introduce Poincaré AdS: it is a hypersurface on $\mathbb{R}^{2,d}$ given by

$$i = 1, ..., d-1 : \begin{cases} X^{-1} &= \dfrac{\ell t}{z} \\ X^0 &= \dfrac{z}{2}\left[1 + \dfrac{1}{z^2}\left(\ell^2 + \sum_{i=1}^{d-1}(x^i)\right)^2 + t^2\right] \\ X^i &= \dfrac{\ell x^i}{z} \\ X^d &= \dfrac{z}{2}\left[1\dfrac{1}{z}\left(\ell^2 - \sum_{i=1}^{d-1}(x^i)^2 - t^2\right)\right] \end{cases} \tag{1.15}$$

This yields the following metric

$$ds_{\text{Poin}}^2 = \frac{\ell^2}{z^2}\left(-dt^2 + dz^2 + \sum_{i=1}^{d-1}(dx^i)^2\right) \tag{1.16}$$

The boundary of AdS in the Poincaré patch appears at $z \to 0$, and we can again strip off the conformal factor.

$$ds_{\text{bdy}}^2 = -dt^2 + \sum_{i=1}^{d-1}(dx^i)^2 = \eta_{\mu\nu}dx^\mu dx^\nu \tag{1.17}$$

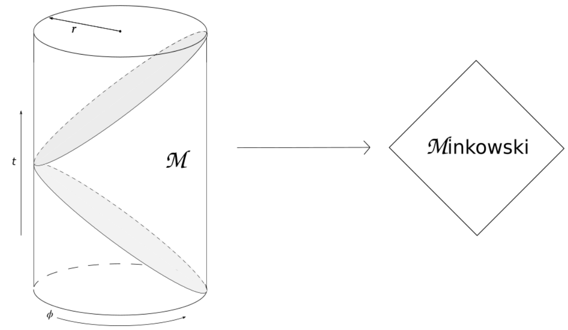


Figure 1.2: Poincaré patch over Global AdS, with its associated d-dimensional Minkowski boundary. The "shaded" areas are known as Poincaré horizons.

Keep in mind that the Poincaré patch covers only half of Global AdS. If instead of stripping off ℓ^2/z^2, we strip off $\ell^2/\lambda^2 z^2$, we end up with

$$ds^2 = \lambda^2 \eta_{\mu\nu} dx^\mu dx^\nu \quad (t \to \lambda^{-1} t, x^i \to \lambda x^i) \tag{1.18}$$

The boundary metric is thus only defined up to a conformal transformation. Now, back to the full (not only boundary) metric, let's take $z = \dfrac{\ell}{r}$. The metric the becomes

$$ds^2_{\text{Poin}} = -\frac{r^2}{\ell} dt^2 + \left(\frac{r^2}{\ell^2}\right)^{-1} dr^2 + \frac{r^2}{\ell^2} \sum_{i=1}^{d-1} (dx^i)^2 \tag{1.19}$$

We can easily observe the similitude with (1.11). For this reason, it is common to find Global AdS called as *spherical slicing*, Poincaré AdS as *flat slicing*, and Rindler AdS

$$ds^2_{\text{Rind}} = -\left(\frac{r^2}{\ell^2} - 1\right) dt^2 + \left(\frac{r^2}{\ell^2} - 1\right)^{-1} dr^2 + r^2\, d\mathbb{H}^2_{d-1} \tag{1.20}$$

as *hyperbolic slicing*. This patch only covers a quarter of Global AdS. A similar image to the one of Poincaré AdS can be found, and again we find a boundary rhombus, however in this case it does not correspond to Minkowski, but to $R_t \times \mathbb{H}_{d-1}$.

There remains to discuss Euclidean AdS, for completeness. It is defined in an analogous way as previously, embedding it this time in proper Minkowski space $\mathbb{R}^{1,d+1}$. It is given by the hyperboloid

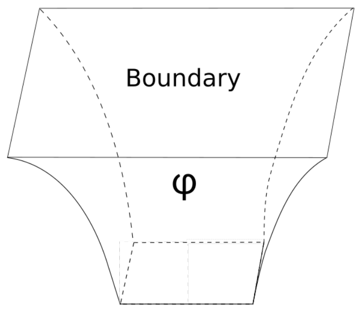
$$(X^{-1})^2 - (X^0)^2 + \sum_{i=1}^{d} (X^i)^2 = -\ell^2 \tag{1.21}$$

We can therefore achieve this by the substitution

$$\begin{cases} X^{-1} & \to iX^{-1} \\ t & \to it_E \end{cases} \Rightarrow \begin{cases} ds^2_{\text{glob}} & = \left(\dfrac{r^2}{\ell^2} + 1\right) dt_E^2 + \left(\dfrac{r^2}{\ell^2} + 1\right)^{-1} dr^2 + r^2\, d\Omega^2_{d-1} \\ ds^2_{\text{Poin}} & = \dfrac{\ell^2}{z^2} \left(dt_E^2 + dz^2 + \displaystyle\sum_{i=1}^{d-1} (dx^i)^2\right) \end{cases} \tag{1.22}$$

It happens that in Euclidean signature, both patches cover the same!

A different way to visualize Poincaré AdS is



Now, if we have a φ scalar field in this setting, what would its dynamics be like?

1.2 Scalar fields in AdS

We have the previous action for the gravitational field S_{EH}, and we add an extra S_{matter}:

$$S_{\text{matter}} = -\int d^{d+1}x\sqrt{-g}\left(\frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi + \frac{m^2}{2}\varphi^2\right) \tag{1.23}$$

The equation of motion derived from this action is not the usual Klein-Gordon equation $(\Box - m^2)\varphi = 0$, as the field is no longer living in flat space; instead this equation is substituted by its analog in curved spacetime:

$$(\nabla^2 - m^2)\varphi = 0, \quad \text{with} \quad \nabla^2 = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) \tag{1.24}$$

In Poincaré AdS (1.16), the action becomes

$$S_{\text{matter}} = -\frac{1}{2}\int d^{d+1}x\left(\frac{\ell}{z}\right)^{d+1}\left[\left(\frac{z}{\ell}\right)^2\left(-\partial_t\varphi\partial_t\varphi + \partial_z\varphi\partial_z\varphi + \sum_{i=1}^{d-1}\partial_i\varphi\partial_i\varphi\right) + \frac{m^2}{2}\varphi^2\right] \tag{1.25}$$

Using as an Ansatz $\varphi(z,t,x') = \left(\frac{z}{\ell}\right)^{d/2}\phi(z,t,x')$,

$$\partial_z\varphi\partial_z\varphi = \left[\frac{d}{2\ell}\left(\frac{z}{\ell}\right)^{d/2-1}\phi + \left(\frac{z}{\ell}\right)^{d/2}\partial_z\phi\right]^2 = \frac{d^2}{4\ell^2}\left(\frac{z}{\ell}\right)^{d-2}\phi^2 + \left(\frac{z}{\ell}\right)^{d}\partial_z\phi\partial_z\phi + \frac{d}{\ell}\left(\frac{z}{\ell}\right)^{d-1}\phi\partial_z\phi$$

With this, the action becomes

$$S_{\text{matter}} = -\frac{1}{2}\int d^{d+1}x\left[\left(\frac{z}{\ell}\right)\left(-\partial_t\phi\partial_t\phi + \sum_{i=1}^{d}\partial_i\phi\partial_i\phi\right) + \underbrace{\cancel{\frac{d}{\ell}\phi\partial_z\phi}}_{\text{total deriv.}} + \frac{z}{\ell}\partial_z\phi\partial_z\phi + \frac{\ell}{z}\frac{d^2}{4\ell^2}\phi^2 + \frac{\ell}{z}m^2\phi^2\right] \tag{1.26}$$

If perform a change of coordinates $y = -\log z$, then

$$S_{\text{matter}} = \frac{1}{2\ell}\int\prod_i dx^i dt dy\left[\ell^2 e^{-2y}(-\partial_t\phi\partial_t\phi + \partial_i\partial_i\phi) + \partial_y\phi\partial_y\phi + \left(m^2\ell^2 + \frac{d^2}{4}\right)\phi^2\right] \tag{1.27}$$

This looks more like a canonical normalization in the y coordinate. We can even go to Fourier space in the t, x^i coordinates and then the first term above becomes $\omega^2 - k^2$. For each k-mode, one then views this as a usual Schrodinger problem, and the ω, k part is like a potential because of the factor of e^{-2y}. Then, as we can see, there has been a shift in the mass:

$$m^2\ell^2 \to m^2\ell^2 + \frac{d^2}{4} \geq 0 \Rightarrow m^2\ell^2 \geq -\frac{d^2}{4} \tag{1.28}$$

This is the so-called Breitenlohner-Freedman bound (BF bound). Note that what we did here is a shortcut and one should really be more careful, but you can look up the references for more details.

Let us now solve Klein-Gordon equation in curved space (1.24). As constant z looks like Minkowski spacetime, it allows for plane wave decomposition in the remaining coordinates. This leads to propose the Ansatz $\varphi(t,z,x) = e^{i(\vec{k}\vec{x}-\omega t)}\varphi(z)$

$$\frac{z^{d+1}}{\ell^{d+1}}\partial_z\left(\frac{\ell^{d-1}}{z^{d-1}}\partial_z\varphi(z)\right) - \left((k^2-\omega^2)\frac{z^2}{\ell^2} + m^2\right)\varphi(z) = 0 \tag{1.29}$$

The solutions to this equation are the Bessel functions. However, as we are interested in the boundary $z \to 0$, the equation near there takes the following form

$$z^{d+1}\partial_z\left(\frac{1}{z^{d-1}}\partial_z\varphi\right) - m^2\ell^2\varphi = 0 \tag{1.30}$$

By power counting in z, we can guess a further Ansatz: $\varphi(z) = z^{\Delta}$. The equation is cast then in the following algebraic form

$$[\Delta(\Delta-d) - m^2\ell^2]z^{\Delta} = 0 \tag{1.31}$$

We have then found that our Ansatz is a solution if and only if $\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\ell^2}$. But, what do these coefficients mean?

Near the boundary, the field behaves as $\varphi(z) \sim z^{\Delta_+} + z^{\Delta_-}$. The normalization can be obtained through the Klein-Gordon inner product:

$$\langle\varphi|\varphi\rangle \sim \int dz\, z^{-d-1}(\varphi(z)\partial_z\varphi(z)) \xrightarrow{z\to 0} \int dz\, z^{-d-1}z^{2\Delta} \sim z^{2\Delta-d} \tag{1.32}$$

This integral converges for $\Delta > d/2$, therefore the two solutions found correspond to

- Δ_+: Normalizable solution, which will be dual to sources $\mathcal{O}_{\Delta}$.
- Δ_-: Non-normalizable solution, which will correspond to VEVs (Vacuum Expectation Values) of the sources

Lecture 2

Anti-de Sitter symmetries and asymptotics

As with any physics problem, one of the first analysis that needs to be carried out is the study of its symmetries, in order to understand better any behaviour of the theory, and even constrain it. In our current case of study, the boundary is something of remarkable relevance, so it is also convenient to analyze it in sufficient detail.

Both subjects will be covered in this chapter, as they constitute important features of Anti-de Sitter spacetimes.

2.1 Symmetries of AdS and conformal transformations

We can infer the symmetries of (Euclidean) AdS from the transformations that preserve the hyperboloid (1.21). These are rotations (preserving $x^2 + y^2$, d+1), and boosts (preserving $x^2 - y^2$, 1): the isometry group is then $SO(1, d+1)$. Translations are not an isometry, due to the presence of ℓ.

The number of Killing vectors (isometries) associated to a maximally symmetric space, like for example AdS, is $\# = (d+1)(d+2)/2$.

Example (Flat spacetime isometries): In usual Minkowski spacetime $\mathbb{R}^{1,3}$, we have the following 10 isometries/Killing vectors:

- 3 rotations
- 3 boosts
- 4 translations

The above formula gives this precise number: $(3+1)(3+2)/2 = 10$

Surprisingly (or not), $SO(1, d+1)$ is also the Euclidean conformal group in d-dimensions! This is one of the first hints about AdS/CFT.

Conformal transformations

You might be wondering what is this all about conformal transformations. The answer is, they are a special type of diffeomorphisms $x^\mu \to x^\mu + \epsilon^\mu(x^\mu)$:

1) Translations: $x^\mu \to x^\mu + a^\mu$

2) Rotations & Boosts: $x^\mu \to \omega^\mu{}_\nu x^\nu$

3) Scalings: $x^\mu \to x^\mu + \lambda x^\mu$

4) Special conformal transformations: $x^\mu \to x^\mu + b^\mu x^\nu x_\nu - 2x^\mu b_\nu x^\nu$

Remark: These are the symmetry properties of the Lagrangian (the theory considered), not the spacetime. Spacetime only has 1), 2).

These symmetries are infinitesimally generated by generators (hence the name):

1) Translations: $P_\mu = -i\partial_\mu$

2) Rotations & Boosts: $M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$

3) Scalings: $D = -ix^\mu \partial_\mu$

4) Special conformal transformations: $K_\mu = i(x^\nu x_\nu \partial_\mu - 2x_\mu x^\nu \partial_\nu)$

The generators satisfy an algebra of the following form

$$\begin{aligned} [D, K_\mu] = iK_\mu, \quad [D, P_\mu] = iP_\mu, \quad [K_\mu, P_\nu] = 2i(\eta_{\mu\nu} D - M_{\mu\nu}) \\ [P_\rho, M_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu), \quad [K_\rho, M_{\mu\nu}] = i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu) \\ [M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}) \end{aligned} \tag{2.1}$$

Now a little discussion about representations. In CFT's the protagonists are the primary operators $\mathcal{O}(0)$. They satisfy

- $[K_\mu, \mathcal{O}(0)] = 0$
- $[D, \mathcal{O}(0)] = i\Delta \mathcal{O}(0)$. The constant Δ is called the scaling dimension, as under a metric rescaling $g \to \Omega^2 g_{\mu\nu}$, the primary scales as $\mathcal{O}(0) \to \Omega^\Delta \mathcal{O}(0)$.
- $[M_{\mu\nu}, \mathcal{O}(0)] = \Sigma_{\mu\nu} \mathcal{O}(0)$. The constant $\Sigma_{\mu\nu}$ is called spin.

Unitarity imposes the bound $\Delta \geq (d-2)/2$. This scaling dimension is of course related to the Δ appearing in the previous section, about scalar fields in AdS.

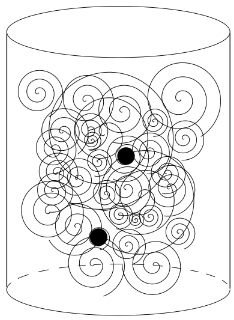
We can obtain new operators (descendants) from primaries by

$$\partial_{\mu_1} \partial_{\mu_2} ... \partial_{\mu_n} \mathcal{O}(0) \tag{2.2}$$

These operators carry a conformal weight (scaling dimension) $\Delta + n$.

It is worth mentioning that in 2d CFT, there are more descendants, $L_{-1}^{k_1} L_{-2}^{k_2} ... \mathcal{O}(0)$.

2.2 Fefferman-Graham coordinates



In "real life", in both quantum/classical gravity, the metric fluctuates, for example in the AdS cylinder. However we are interested in preserving the symmetries, at least in the boundary (processes far away from sources). We therefore need some type of boundary condition, but without being too restrictive, as we want to allow a variety of things to happen in the interior, such as black holes, galaxies, etc.

2.2.1 Asymptotically AdS spacetimes

An Asymptotically AdS spacetime (AAdS):

1. Is a solution to the Einstein's equations

2. Preserves the conformal structure at the boundary:

$$g_{\mu\nu} \sim \frac{r^2}{\ell^2} g^{(0)}_{\mu\nu} + O(r^2) g^{(1)}_{\mu\nu} + ... \sim \frac{\ell^2}{z^2} g^{(0)}_{\mu\nu} + O\left(\frac{1}{z^2}\right) g^{(1)}_{\mu\nu} + ... \tag{2.3}$$

   The first term is "Pure AdS", while the latter corresponds to the fluctuations inside (corrections).

   More precisely, let $\Omega(x^\mu)$ have a single zero near the boundary, then

$$\text{AAdS:} \quad \lim_{\text{bdy}} \to \Omega^2 g_{\mu\nu} \quad \text{has a well defined conformal structure.} \tag{2.4}$$

Example (Poincaré and Global):

- Poincaré: boundary when $z \to 0 \Rightarrow \Omega = f(t, x^i) z$

$$\lim_{z\to 0} \Omega^2 g_{\mu\nu} = \lim_{z \to 0} f^2(t, x^i) z^2 \frac{\ell^2}{z^2}(-dt^2 + dz^2 + ...) \propto f^2(t, x^i) \left(-dt^2 + \sum_{i=1}^{d-1} (dx^i)^2\right) \tag{2.5}$$

- Global: boundary when $\rho \to \infty \Rightarrow \Omega = f(x^i) \frac{1}{e^\rho}$.

$$\lim_{\rho \to \infty} \Omega^2 g_{\mu\nu} = f(t, x^i)^2 (-dt^2 + d\Omega^2_{d-1}) \tag{2.6}$$

Counterexample: $ds^2 = ds^2_{\text{AdS}} + r^4 \tilde{g}_{\mu\nu}(t, x^i) \to \Omega = f(t, x^i)/r$

$$\lim_{r \to \infty} \Omega g_{\mu\nu} \sim r^2 \tilde{g}_{\mu\nu}(x^i) \tag{2.7}$$

Let us work out in some detail a less trivial example: AdS_4 Schwarzschid black hole:

$$ds^2 = -\left(\frac{r^2}{\ell^2} + 1 - \frac{2M}{r}\right)dt^2 + \left(\frac{r^2}{\ell^2} + 1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_{d-1}^2 \tag{2.8}$$

$$\approx -\frac{r^2}{\ell^2}dt^2 + \left(\frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\Omega_{d-1}^2 - \underbrace{\left(1 - \frac{2M}{r}\right)dt^2 + \left(-\frac{1}{r^4} + \frac{2M}{r^5}\right)dr^2}_{\text{Subleading}} \tag{2.9}$$

We see that at $r \to \infty$ we do recover AdS_4.

Expanding g_{tt},

$$g_{tt} = -\frac{r^2}{\ell^2}\left(1 + \frac{1}{r^2} - \frac{2M\ell^2}{r^3}\right) \tag{2.10}$$

With the first factor we recover, at large enough r, a boundary metric $ds^2_{\text{bdy}} = -dt^2 + d\Omega^2_{d-1}$. What about the other two terms? Recall the scaling dimensions $\Delta_\pm$ found from (1.31). If $d = 3$,

$$1 \sim \frac{1}{r^{\Delta_-}} \sim \frac{1}{r^0} \quad \text{Source} \tag{2.11}$$

$$-\frac{2M\ell^2}{r^3}\frac{1}{r^{\Delta_+}} \sim \frac{1}{r^3} \quad \text{VEV} \tag{2.12}$$

These two terms lead to independent solutions near the boundary. The $1/r^2$ is just a correction to these solutions. Therefore,

$$\phi \sim z^{\Delta_+}(1 + z^2 + ...) + z^{\Delta_-}(1 + z^2...) \tag{2.13}$$

In this case, $\Delta_+ = d$ and the CFT operator is no other than the stress-tensor.

As we've seen already, AAdS boundary conditions allow for a rich space of solutions. We will now discuss this in more detail.

Comment: Defined Ω, which had a single zero at the boundary, the boundary metric being $ds^2_{\text{dby}} = \lim_{z\to 0}\Omega^2(x^i, z)g_{\mu\nu}dx^\mu dx^\nu$. We can further define $\Omega^z = zf(x^i)$ or $\tilde{\Omega} = z\tilde{f}(x^i)$, and then

$${ds^2_{\text{bdy}}}^{(1)} = f^2(x^i)\eta_{\mu\nu}dx^\mu dx^\nu \tag{2.14}$$

$${ds^2_{\text{bdy}}}^{(2)} = \tilde{f}^2(x^i)\eta_{\mu\nu}dx^\mu dx^\nu \tag{2.15}$$

As we can see, there is no unique boundary metric, it is defined up to a conformal transformation!

$$\text{Weyl}: \quad ds^{2(1)} \to \frac{\tilde{f}^2(x^i)}{f^2(x^i)}ds^{2(1)} = ds^{2(2)} \tag{2.16}$$

Fefferman-Graham Coordinates

If we take $\rho = z^2$ (careful! this is not the same ρ as before), then

$$ds^2_{\text{AdS}} = \ell^2\left[\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\left(-dt^2 + \sum_{i=1}^{d-1}(dx^i)^2\right)\right] \tag{2.17}$$

Fefferman and Graham showed [7] that any asymptotic AdS metric can be written as

$$ds^2 = \ell^2 \left(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{\mu\nu}(\rho, x^\mu) dx^\mu dx^\nu \right) \tag{2.18}$$

Not only they showed this, but they also did show that

$$g_{\mu\nu}(x^\mu, \rho) = g^{(0)}_{\mu\nu}(x^\mu) + \rho g^{(2)}_{\mu\nu}(x^\mu) + ... + \rho^{d/2} g^{(d)}_{\mu\nu}(x^\mu) + ... + \underbrace{h^{(d)}_{\mu\nu}(x^\mu)\rho^{d/2}\log\rho}_{\text{in even dimension d}} \tag{2.19}$$

Given $g^{(0)}_{\mu\nu}$, using Einstein's equations, $g^{(2)}_{\mu\nu}, g^{(4)}_{\mu\nu}, ..., g^{(d-2)}_{\mu\nu}$ and $h_{\mu\nu}$ are all fixed. The only one non fixed is $g^{(d)}_{\mu\nu}$, only its trace is fixed. However, requiring regularity in the interior $\rho \to \infty$, $g^{(d)}_{\mu\nu}$ is also fixed. (This makes sense, as the Einstein's equations are 2nd order differential equations, therefore we need two conditions to fix the solution: $g^{(0)}_{\mu\nu}$, and regularity at $\rho \to \infty$)

If (in $d > 2$) the boundary is flat, $g(x^\mu, \rho) = g^{(0)} + \rho g^{(2)} + \rho^2 g^{(4)} + ...$

$$g^{(2)}_{\mu\nu} = \frac{1}{d-2}\left(R^{(0)}_{\mu\nu} - \frac{R^{(0)}}{2(d-1)} g^{(0)}_{\mu\nu} \right), \quad g^{(4)}_{\mu\nu} = \frac{1}{4}\left(g^{(2)}_{\mu\nu}\right)^2 \tag{2.20}$$

What happens if $g^{(0)}_{\mu\nu} \to g^{(0)}_{\mu\nu} e^{2\sigma(x)}$? Consider a change of coordinates:

$$\rho = \rho' e^{-2\sigma(x^\mu)} + \sum_{k=2} a_k(x^\mu)(\rho')^k \tag{2.21}$$

$$x^\mu = (x')^\mu + \sum_{k=1} a^\mu_k(x^\mu)(\rho')^k \tag{2.22}$$

The line element changes as

$$ds^2 = \ell^2 \left(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{\mu\nu}(\rho, x^\mu) dx^\mu dx^\nu \right) \approx \ell^2 \left(... + \frac{e^{2\sigma(x^\mu)}}{\rho'} g_{\mu\nu}(\rho, x^\mu) dx^\mu dx^\nu \right) + ... \tag{2.23}$$

One can check that this preserves the F-G gauge. It uniquely fixes the functions a

For example:

$$a_2 = -\frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma)e^{-4\sigma} \quad a_3 = \frac{1}{4}e^{-6\sigma}\left(\frac{3}{4}(\partial\sigma)^2 + \partial^\mu\sigma\partial^\nu\sigma g^{(2)}_{\mu\nu}\right)$$
$$a^\mu_1 = \frac{1}{2}\partial^\mu\sigma e^{-2\sigma} \quad a^\mu_2 = -\frac{1}{4}e^{-4\sigma}\left(\partial_k\sigma g^{\mu k(2)} + \frac{1}{2}\partial^\mu\sigma(\partial\sigma)^2 + \frac{1}{2}\Gamma^{\mu\,(2)}_{kl}\partial^k\sigma\partial^l\sigma\right) \tag{2.24}$$
$$...$$

By doing a change of coordinates (diffeomorphism), we induced a conformal transformation on the boundary. There are, however two classes of diffeomorphisms:

1. Die fast at the boundary: $\rho = \cancel{\rho' e^{-2\sigma}} + \sum_k (\rho')^2 ...,\quad x = x + \sum \rho....$ These ones do not affect the boundary.

$$\rho = (\rho')^2 \to \text{decays faster.}$$
$$g^{(2)}_{\mu\nu} \to g^{(0)}_{\mu\nu}$$

2. Die slow diffeos: $\rho = \rho' e^{-2\sigma}$. These induce a Weyl transformation. Let's see how in the Poincaré to Global case:

$$ds^2_{\text{bdy},\mathbb{R}^d} = dt_E^2 + \sum_{i=1}^{d-1}(dx^i)^2 \xrightarrow{Polar} dr^2 + r^2 d\Omega^2_{d-1} \tag{2.25}$$

Under a change of coordinate $r = \ell e^\tau, dr = \ell e^\tau d\tau$:

$$ds^2_{\mathbb{R}^d} = e^{2\tau}\left(d\tau^2 + d\Omega^2_{d-1}\right) = e^{2\tau}(ds^2_{\mathbb{R}\times S_{d-1}}) \tag{2.26}$$

Let's try to go the other way around. Now, $e^{-2\tau}(g_{\mu\nu}dx^\mu dx^\nu) = \ell^2(d\tau^2 + d\Omega^2_{d-1})$. Poincaré AdS is then

$$ds^2_{\text{Poin}} = \ell^2\left(\frac{d\rho^2}{4\rho^2} + \frac{e^{2\tau}}{\rho}(d\tau^2 + d\Omega^2_{d-1})\right) \tag{2.27}$$

Using the Fefferman-Graham expansion, (2.21), we arrive to

$$ds^2 = \ell^2\left(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho'}(d\tau^2 + d\Omega^2_{d-1}) + O(\rho^0)\right) \tag{2.28}$$

We will keep this result in standby now, and write the Global AdS metric (1.11) in the F-G gauge. For that,

$$\frac{dr}{\sqrt{\frac{r^2}{\ell^2}+1}} = \ell\frac{d\rho}{2\rho} \Rightarrow \operatorname{arcsinh}\left(\frac{r}{\ell}\right) = \frac{1}{2}\log(\rho/4) \Rightarrow r = \ell^2\frac{1-\frac{\rho}{4\ell^2}}{\sqrt{\rho}} \tag{2.29}$$

Plugging this in the Global AdS metric we find

$$ds^2 = \frac{\ell^4}{\rho}\left(d\tau^2 + \ell^2 d\Omega^2_{d-1}\right) + \frac{\ell^2}{2}d\tau - \frac{\ell^2}{2}d\Omega^2_{d-1} + ... \tag{2.30}$$

We find that (not so surprisingly) there is no obvious mismatch between what we just found for the Poincaré case and this result.

2.3 Exercises (Lectures 1 & 2)

Problem 1: Embedding Coordinates

This problem will help you familiarize yourself with AdS as a hypersurface and embedding coordinates. Consider the metric in $\mathbb{R}^{2,4}$

$$ds^2 = -(dX^{-1})^2 - (dX^0)^2 + \sum_{i=1}^{4}(dX^i)^2 \,. \tag{2.31}$$

We saw in class that AdS_5 is the hypersurface

$$-(X^{-1})^2 - (X^0)^2 + \sum_{i=1}^{4}(X^i)^2 = -\ell^2 \tag{2.32}$$

The metric of the Rindler patch of AdS_5 is

$$ds^2 = -(r^2-\ell^2)dt^2 + \frac{dr^2}{\frac{r^2}{\ell^2}-1} + r^2 d\mathbb{H}_3^2 \,, \tag{2.33}$$

with

$$d\mathbb{H}_3^2 = du^2 + \sinh^2 u(d\theta^2 + \sin^2\theta d\varphi^2) \,. \tag{2.34}$$

Find the embedding coordinates $X^\mu(t,r,u,\theta,\phi)$ corresponding to this patch and derive the metric (2.33) using the formula for the induced metric.

As a first step to solve this problem, first find the embedding coordinates that give the metric

$$ds^2 = \ell^2\left(-\sinh^2\rho dt^2 + d\rho^2 + \cosh^2\rho d\mathbb{H}_3^2\right) \,. \tag{2.35}$$

Then, perform a simple coordinate change between ρ and r.

Problem 2: Klein-Gordon equation in global AdS

Consider the metric on global AdS_4:

$$ds^2 = \ell^2(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho(d\theta^2 + \sin^2\theta d\varphi^2)) \,. \tag{2.36}$$

We will solve the Klein-Gordon equation on this background. The KG equation is

$$(\nabla^2 - m^2)\phi = 0 \,. \tag{2.37}$$

Take for an ansatz:

$$\phi(t,\rho,\theta,\varphi) = e^{-i\omega t}Y_{l,m}(\theta,\varphi)R_{\omega,l,m}(\rho) \,, \tag{2.38}$$

where the $Y_{l,m}$ are the usual spherical harmonics on the 2-sphere. Proceed in the following steps

1. Write the Laplacian explicitly, and apply the time and angular derivatives. You will need to use that the $Y_{l,m}$ are eigenfunctions of the laplacian for the 2-sphere with eigenvalues $-l(l+1)$. At the end, you will have an ODE just for the function $R_{\omega,l,m}(\rho)$.

2. Solve this ODE. You may use Mathematica or another program for this problem, without it it will be hard.

3. Since it is a second order differential equation, it has two solutions. First, we will expand near the origin $\rho = 0$. You need to require that the solution is regular there. One of the two solutions blows up at the origin so this will fix one of the integration constants.

4. Now take this solution and expand it near the boundary $\rho = \infty$. You will get a linear superposition of $e^{-\Delta_+ \rho}$ and $e^{-\Delta_- \rho}$. We need to make sure we only keep the normalizable solution that goes like Δ_+. Since the remaining integration constant is an overall factor in front of the function, it cannot help with this. This means you need to do something more. For this to work, you will need to set ω to be a function of the mass and l. Remember that Γ functions diverge if their argument is a non-positive integer.

5. Show that you find a discrete spectrum of frequencies:

$$|\omega| = \frac{3}{2} + \sqrt{\frac{9}{4} + m^2\ell^2} + l + 2n \tag{2.39}$$

for n a non-negative integer. This is a property of AdS! Even if its infinite, it acts like a "box" and makes the frequencies discrete. Rewriting in terms of the conformal dimension Δ, we find

$$|\omega| = \Delta + l + 2n \tag{2.40}$$

They are integer space, what does this correspond to in terms of some CFT operator $\mathcal{O}$? What does increasing l and n do?

Problem 3: From Poincare to Global

In this problem, we will familiarize ourself with the Fefferman-Graham expansion. Some of this will review what has been done in class (Watch out, typos in class), and go beyond. Start with the Poincare metric in FG form:

$$ds^2 = \frac{\ell^2 d\rho^2}{4\rho^2} + \frac{\ell^2}{\rho}(dt_E^2 + \sum_{i=1}^{d-1} dx_i^2) = \frac{\ell^2 d\rho^2}{4\rho^2} + \frac{\ell^4}{\rho} e^{2\tau}(d\tau^2 + d\Omega_{d-1}^2)\,. \tag{2.41}$$

Now consider the global AdS metric and put it in FG form. You find

$$ds^2 = \ell^2\left[\frac{d\rho^2}{4\rho^2} + \frac{\ell^2}{\rho}(d\tau^2 + d\Omega_{d-1}^2) + \frac{1}{2}(d\tau^2 - d\Omega_{d-1}^2) + ...\right] \tag{2.42}$$

Note that I have used a global time coordinate that is dimensionless, so the metric in the r coordinate would be

$$ds^2 = (r^2 + \ell^2)d\tau^2 + \frac{dr^2}{\frac{r^2}{\ell^2} + 1} + r^2 d\Omega_{d-1}^2 \tag{2.43}$$

The answer (2.42) is written as a power series expansion in ρ, and we keep the terms up to the constant terms in ρ. Now we will perform the conformal transformation on the boundary metric, and compare it to the diffeomorphism in the bulk. This is the coordinate transformation we saw in class

$$\rho = \rho' e^{-2\sigma(x')} + \sum_{k=2} a_{(k)}(x') {\rho'}^k \tag{2.44}$$

$$x^i = x^{i'} + \sum_{k=1} a^i_{(k)}(x') {\rho'}^k \tag{2.45}$$

with the first two terms being

$$a_{(2)} = -\frac{1}{2}(\partial_i \sigma \partial^i \sigma) e^{-4\sigma} \tag{2.46}$$

$$a^i_{(1)} = \frac{1}{2}\partial^i \sigma e^{-2\sigma} \tag{2.47}$$

In this case, we have

$$\sigma(x') = -\tau' \tag{2.48}$$

Note that all derivatives in (2.46) are with respect to the prime coordinates. To lower/raise indices, we use the metric $g^{(0)}_{ij}(x')$ (note the prime here!) which in this case is

$$g^{(0)}_{ij}(x) dx^i dx^j = e^{2\tau}(d\tau^2 + d\Omega^2_{d-1}) \tag{2.49}$$

Taking the poincare metric (2.41) and applying this coordinate transformation (2.44), show that you can generate both the boundary metric $\mathbb{R} \times S^{d-1}$, as well as the constant terms in the expansion of the global metric. Don't forget to Taylor expand your result after doing the coordinate change (you may want to use mathematica to implement the change of coordinates). When doing this, you will see that the ρ-part of the metric is

$$d\rho^2 \left(\frac{\ell^2}{4\rho^2} - \frac{3}{16\ell^2} \right) + \ldots \tag{2.50}$$

so it is not in FG form. Can you guess why this is happening? What did we not take into account?

Lecture 3

Introduction to supersymmetry

Concrete examples of AdS/CFT are best understood when the CFT is supersymmetric. It is possible that the duality holds more generally but due to the nature of the strong/weak relation between both theories, making explicit checks is very hard without supersymmetry, as it gives one a great control over the theory (i.e. protected quantities).

Not only it is useful for the conjecture, but it is also an interesting subject on its own right, and in fact not all supersymmetric theories are expected to have weakly coupled gravity duals.

In order to motivate this part of the course, we are going to begin studying the Wess-Zumino model, which will lead us naturally to the SUSY algebra, and all the machinery behind it.

Motivation: the Wess-Zumino model

Consider the following action, consisting on a free theory for a scalar field A, a pseudoscalar B, and a Majorana spinor χ (we will define them later in the chapter):

$$\mathcal{L}_{\mathrm{WZ}} = -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}\bar{\chi}\partial\!\!\!/\chi; \quad \partial\!\!\!/ \equiv \gamma^\mu \partial_\mu \tag{3.1}$$

What symmetries does the theory posses? It is clear that it is Poincaré invariant[1], but it does also have a hidden symmetry: the Lagrangian is invariant (up to boundary terms) under the following, the most general, supersymmetric transformation:

$$\begin{cases} \delta A &= \bar{\epsilon}\chi \\ \delta B &= i\bar{\epsilon}\gamma_5\chi \\ \delta\chi &= \left[\partial\!\!\!/(A + i\gamma_5 B)\right]\epsilon \end{cases} \tag{3.2}$$

Here it can be seen the difference between the scalar and the pseudoscalar; they have different transformation laws, the pseudoscalar one will depend on the chirality matrix. It is interesting to notice that the on-shell degrees of freedom of the two bosonic fields A, B is equal to the ones of the Majorana fermion χ; this is no coincidence, as supersymmetry requires same number of bosons and fermion.

[1]It might seem that there is also a global U(1) symmetry for χ, however as we will see later the fact that they are Majorana is a reality condition, hence it can not be charged under this symmetry. If there were more of these fields, χ^I, that could be a possibility, rotating them.

Understanding these sets of transformations is going to be the subject of this chapter. We will come back to this model later on, but in order to successfully do so, we need to develop the notions and technology behind supersymmetry.

3.1 Symmetries of the S-matrix, and no-go's

First of all, we should understand what kind of symmetry is supersymmetry. Let us leave that aside for a moment, and study usual QFT symmetries. It is well known that they can be divided into two categories:

1. Spacetime symmetries (such as ISO$(d,1)$, i.e. Poincaré) which are global.
2. Internal symmetries. These symmetries can either be global (the baryonic symmetry $\mathrm{U}(1)_B$, with an associated conserved charge, the baryonic number) or local (such as the SU(2)× U(1) sector of the Standard Model, unifying electromagnetism and the weak force).

An important question is: can we unify spacetime and internal symmetries? It could be wonderful if we could have a group that unifies them.

"No-go" theorem (Coleman-Mandula, 1967): The most general symmetry group of the S-matrix is

$$G = G_{\text{Poincare}} \times G_{\text{internal}} \tag{3.3}$$

This doesn't allow for unification. However, this theorem only takes bosonic generators into account.

The statement of the "No-go" theorem is also written as

$$[T_a, P_\mu] = [T_a, M_{\mu\nu}] = 0 \tag{3.4}$$

where $P_\mu, M_{\mu\nu}$ are the generators of translations and rotations, respectively, and T_a are the generators of the internal symmetries (satisfying a Lie algebra $[T_a, T_b] = f_{ab}{}^c T_c$).

Allowing for the possibility of anticommutation relations, a new theorem was found:

"Yes-go" theorem (Haag-Lopuszanski-Sohnius, 1975): The most general symmetry of the S-matrix is

$$G = G_{\text{SuperPoincare}} \times G_{\text{internal}} \tag{3.5}$$

where $G_{\text{SuperPoincare}}$ is a unique non-trivial extension of Poincaré, including anticommutating generators Q.

3.2 Superalgebras

Recall that, under an infinitesimal symmetry transformation

$$\delta_\varepsilon X = (\varepsilon T)X, \qquad X = \{\text{fields}\} \tag{3.6}$$

with ε the parameter of the transformation, and T the generator. This εT combination has to be bosonic for being a symmetry, hence

$$\begin{cases} T_a & \text{bosonic} & \to \varepsilon_a & \text{bosonic} \\ Q_\alpha & \text{fermionic} & \to \varepsilon_\alpha & \text{fermionic} \end{cases}$$

A superalgebra for (T_a, Q_α) is then given by:

$$[T_a, T_b] = f_{ab}{}^c\, T_c; \quad \{Q_\alpha, Q_\beta\} = \tilde{f}_{\alpha\beta}{}^c\, T_c; \quad [T_a, Q_\beta] = \hat{f}_{a\beta}{}^\gamma\, Q_\gamma \tag{3.7}$$

where $\tilde{f}_{\alpha\beta}{}^c$ and $\hat{f}_{a\beta}{}^\gamma$ are a new set of structure constants. Let us now introduce the notation and tools we will be using in what follows.

Spinor technology (4d)

The algebra associated to spinors is the Clifford algebra:

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}, \quad \mu, \nu = 0, ..3 \tag{3.8}$$

It is usual to also define the chirality matrix $\gamma^5 \equiv -i\gamma_0\gamma_1\gamma_2\gamma_3$, satisfying $(\gamma^5)^2 = \mathbb{I}$.

Dirac spinors will be ψ_α, and the gamma matrices act on them by $\bar{\psi}_\alpha = (\gamma_\mu)_\alpha{}^\beta \psi_\beta$.

The charge conjugation matrix is C, such that $C\gamma^\mu C^{-1} = -(\gamma^\mu)^T$, and this matrix satisfies $C = -C^T$.

There are two different notions of conjugation:

- Dirac conjugation: $\bar{\psi}_D \equiv = \psi^\dagger i\gamma^0 \Longleftrightarrow \bar{\psi}_D^\alpha = \psi_\beta^*(i\gamma^0)^\alpha{}_\beta$
- Majorana conjugation: $\bar{\psi}_M \equiv \psi^T C \Longleftrightarrow \bar{\psi}_M^\alpha = \psi_\beta C^{\beta\alpha}$

  A Majorana fermion, by definition, is a Dirac spinor satisfying the reality condition $\bar{\psi}_D = \bar{\psi}_M$

We will also need the following properties of Majorana fermions ε, χ (see 4.4, Problem 2):

$$\begin{aligned} \bar{\varepsilon}\chi = \bar{\chi}\varepsilon; \quad \bar{\varepsilon}\gamma_5\chi = \bar{\chi}\gamma_5\varepsilon; \quad \bar{\varepsilon}\gamma^\mu\chi = -\bar{\chi}\gamma^\mu\varepsilon \\ \bar{\varepsilon}\gamma_\mu\gamma_5\chi = \bar{\chi}\gamma_\mu\gamma_5\varepsilon; \quad \bar{\varepsilon}\gamma_{\mu\nu}\chi = -\bar{\chi}\gamma_{\mu\nu}\varepsilon \end{aligned} \tag{3.9}$$

where $\gamma_{\mu\nu} \equiv \dfrac{1}{2}[\gamma_\mu, \gamma_\nu]$.

SuperPoincaré algebra

The ($\mathcal{N} = 1$) SuperPoincaré algebra has for generators $(Q_\alpha, P_\mu, M_{\mu\nu})$, where the $P_\mu, M_{\mu\nu}$ satisfy the usual Poincaré algebra. The remaining (anti-)commutation relations between the generators are listen below, with their implications

- Supersymmetry is a spacetime symmetry, applied twice generates a translation:

$$\{Q_\alpha, Q_\beta\} = 2(\gamma_\mu C)_{\alpha\beta} P^\mu \tag{3.10}$$

- Fermions and bosons have the same mass:

$$[Q_\alpha, P_\mu] = 0 \Rightarrow [Q_\alpha, P_\mu P^\mu] = 0 \tag{3.11}$$

- Q's transform as spinors under the Lorentz group:

$$[Q_\alpha, M_{\mu\nu}] = \frac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta \tag{3.12}$$

- There is an internal generator, R, which is a group automorphism (that is, although being a generator, it is never generated in the RHS). In this case it is called an U(1) R-symmetry

$$[Q_\alpha, R] = i(\gamma_5)_\alpha{}^\beta Q_\beta \tag{3.13}$$

Claim: $\mathcal{N}=1$ superPoincaré is the unique non-trivial extension of Poincaré with one Q_α.

Proof: It is clear that, if F means fermion and B means boson, then $\{F,F\}=B$, $[B,B]=B$, $[F,B]=F$. The proof then follows from the "superjacobian" identities, and the Majorana condition.

Fun with Jacobi identities (see 4.4, Problem 1):

$$\begin{aligned}
[[B_1,B_2],B_3]+[[B_3,B_1],B_2]+[[B_2,B_3],B_1]&=0\\
[[B_1,B_2],F_3]+[[F_3,B_1],B_2]+[[B_2,F_3],B_1]&=0\\
\{[B_1,F_2],F_3\}-\{[F_3,B_1],F_2\}+[\{F_2,F_3\},B_1]&=0\\
[\{F_1,F_2\},F_3]+[\{F_3,F_1\},F_2]+[\{F_2,F_3\},F_1]&=0
\end{aligned} \tag{3.14}$$

It can be shown that with the ansatz $[Q_\alpha, P_\mu]=(c\gamma_\mu + d\gamma_\mu\gamma_5)_\alpha{}^\beta Q_\beta$, the superjacobian implies $c,d=0$. The proof follows from a generalization of this, for all (anti-)commutators, writing the most general (anti-)commutator as the most general one, and checking it with the superjacobian. □

Immediate consequences of the SUSY algebra

Four important properties follow from our discussion:

1. Q_α is in the spin 1/2 representation of the Lorentz algebra. Thus, given a state with angular momentum j,

$$|j\rangle \overset{Q_\alpha}{\longleftrightarrow} |j\pm 1/2\rangle \tag{3.15}$$

SUSY then mixes bosonic and fermionic fields. Another point of view (that we won't discuss here) is that Q_α generate translations in Superspace.

2. From (3.11), it is clear that $P^2=-m^2$ is a Casimir operator of the SUSY algebra, hence

$$|j,m\rangle \overset{Q_\alpha}{\longleftrightarrow} |j\pm 1/2, m\rangle \tag{3.16}$$

(where m is the mass label).

3. Energy $E\equiv iP_0$ is positive semi-definite, $E\geq 0$.

Proof: This follows from $\{Q_\alpha, Q_\beta\}=2(\gamma_\mu C)_{\alpha\beta}P^\mu$. Multiplying times $C^{\delta\beta}$, we obtain

$$\begin{aligned}
&(Q_\alpha Q_\beta + Q_\beta Q_\alpha)C^{\delta\beta} = 2(\gamma_\mu C)_{\alpha\beta}C^{\delta\beta}P^\mu\\
&\Rightarrow \{Q_\alpha, \bar{Q}^\delta\} = 2C^{\delta\beta}(\gamma_\mu)_\alpha{}^\gamma C_{\gamma\beta}P^\mu = -2\delta^\delta{}_\gamma(\gamma_\mu)_\alpha{}^\gamma P^\mu = -2(\gamma_\mu)_\alpha{}^\delta P^\mu
\end{aligned}$$

If we now multiply by $i\gamma^0$ and trace over the spinor indices:

$$\mathrm{Tr}_{\alpha,\beta}\left[\{Q_\alpha,\bar{Q}^\delta\}i(\gamma^0)_\delta{}^\beta\right] = -2\,\mathrm{Tr}_{\alpha,\beta}\left[(\gamma_\mu)_\alpha{}^\delta(i\gamma^0)_\delta{}^\beta P^\mu\right]$$

Using cyclicity of the trace, plus the identity $\mathrm{Tr}\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \mathrm{Tr}\,\mathbb{I} = 8\eta_{\mu\nu}$, the RHS reads

$$-2i\,\mathrm{Tr}_{\alpha,\beta}\left[(\gamma_\mu\gamma^0)_\alpha{}^\beta\right]P^\mu = -i\,\mathrm{Tr}\left[(\gamma_\mu\gamma^0 + \gamma^0\gamma_\mu)_\alpha{}^\beta\right]P^\mu = -8i\delta^0{}_\mu P^\mu = 8iP_0$$

While the LHS reads, using $\bar{\psi}_D = \psi^\dagger i\gamma^0 \Rightarrow \bar{\psi}_D i\gamma^0 = \psi^\dagger$.

$$\mathrm{Tr}_{\alpha,\beta}\left[Q_\alpha\bar{Q}^\delta(i\gamma^0)_\alpha{}^\beta + \bar{Q}^\delta(i\gamma^0)_\delta{}^\beta Q_\alpha\right] = \mathrm{Tr}_{\alpha,\beta}\left[Q_\alpha Q^{\dagger\beta} + Q^{\dagger\beta}Q_\alpha\right]$$

Therefore, $E = iP_0 \propto Q_\alpha Q^{\dagger\alpha} + h.c. \geq 0\,\square$.

In particular, the vacuum state has $E_{vac} = 0$ stable! Quantum corrections don't change this if the vacuum is supersymmetric. This is one example of a quantity protected by supersymmetry.

4. The number of fermions and bosons is the same:

Theorem: In any representation of SUSY in which P_μ is a one-to-one operator, there are equal number of fermionic and bosonic degrees of freedom.

This is a property of the algebra, and holds both on-shell and off-shell (this will be important in what follows). Proof by picture: Suppose that it is not the case, then

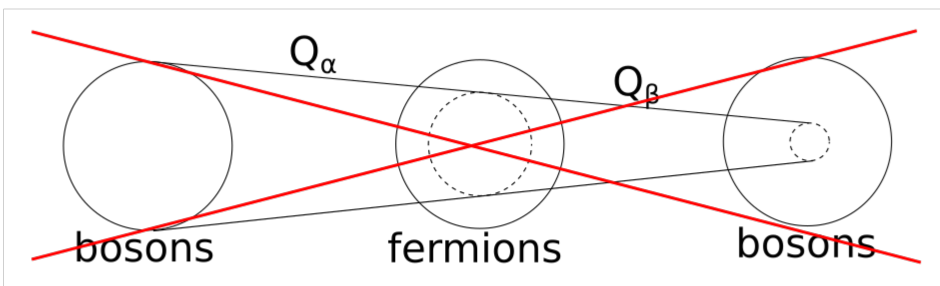


Figure 3.1: Two Q's are a translation ($\{Q, Q\} \sim P$ is one-to-one), hence this cannot happen. $\square$

Counting degrees of freedom

There are two types of counting when it comes to degrees of freedom:

- On-shell d.o.f: Number of helicity states (physical/propagating degrees of freedom)
- Off-shell d.o.f: Number of field field components minus gauge transformations.

This is exemplified with a table of some commonly discussed fields (Table 3.1), where $[x]$ refers to the integer part of x. Restricting to $d = 4, s \leq 1$ we have (Table 3.2).

We can see that there are two basic 4d $\mathcal{N} = 1$ on-shell multiplets (which means that they contain the same amount of fermions and bosons, which is required by supersymmetry):

- Chiral multiplet ("matter"): $(\underbrace{\chi_\alpha}_{\text{2 ferm.}}, \underbrace{A, B}_{\text{2 bos.}}) + (F, G)$

Field	Off-shell	On-shell
Real scalar	1	1
Dirac spinor	$2^{\left[\frac{d}{2}\right]}$ complex	$\frac{1}{2}2^{\left[\frac{d}{2}\right]}$ complex
Spin-1 A_μ	$d-1$	$d-2$
Spin-3/2 ${\psi_\mu}^\alpha$	$(d-1)2^{\left[\frac{d}{2}\right]}$ complex	$\frac{1}{2}(d-3)2^{\left[\frac{d}{2}\right]}$ complex
Graviton $g_{\mu\nu}$	$\frac{1}{2}d(d-1)$	$\frac{1}{2}(d-1)(d-2)-1$

Table 3.1: Some degrees of freedom for general d, s

Field	Off-shell	On-shell
Real scalar A	1	1
Pseudoscalar B	1	1
Majorana spinor χ_α	4	2
Spin-1 A_μ	3	2

Table 3.2: Degrees of freedom with $d=4, s \leq 1$ interesting for us in what follows

- Vector multiplet: $(\lambda_\alpha, A_\mu) + D$

where F, G, D are auxiliary fields for off-shell SUSY. The chiral multiplet will constitute the Wess-Zumino model, which made its appearance at the beginning of this chapter, whereas the vector multiplet is SuperQED if $G=$U(1), or SuperQCD if $G=$SU(N_c). For now we will only focus on the on-shell multiplets.

Extended SUSY

We can generalize the SuperPoincaré algebra to one with generic $\mathcal{N}$. This number indicates the number of supercharges, Q^i_A, $i=1,...,\mathcal{N}$. The algebra gets a bit more messier because of the new indices, but not much:

$$\begin{aligned} \{Q^i_A, \bar{Q}_{Bj}\} &= -2i(\sigma^\mu)_{AB}{\delta^i}_j P_\mu; \quad \{Q^i_A, Q^j_B\} = \epsilon_{AB}(U^{ij}+iV^{ij}); \\ [Q^i_A, M_{\mu\nu}] &= \frac{1}{2}(\sigma_{\mu\nu})^B{}_A Q^i_B; \quad [Q^i_A, T_R] = {R^i}_j Q^j_A \end{aligned} \tag{3.17}$$

where $U^{ij}=-U^{ji}$, $\quad V^{ij}=-V^{ji}$ are the "central charges", as $[\bullet, U]=[\bullet, V]=0$, and $R=-R^T$ (U$(\mathcal{N})_R$ is the larger R-symmetry, mixing the i,j indices). This algebra goes by the name of $\mathcal{N}$-extended SuperPoincaré algebra.

3.3 Wess-Zumino model

Finally, we have the tools to analyze properly the Wess-Zumino model, shown at the beginning of this chapter (3.1). It is the correct guess for the action of the on-shell $\mathcal{N}=1$ chiral multiplet, enjoying Poincaré symmetry along with the supersymmetric transformations (3.2). We will now show the closure of these transformations, its invariance up to boundary terms, and finally we will briefly mention the generalization of this model which includes interactions.

Closure of SUSY algebra

We have to prove that the set of transformations defined above does indeed satisfy the supersymmetry algebra. In other words, does $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]$ close to the SUSY algebra?

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]A = \delta_{\epsilon_1}(\bar{\epsilon_2}\chi) - \delta_{\epsilon_2}(\bar{\epsilon_1}\chi) = \bar{\epsilon_2}\partial\!\!\!/(A + i\gamma_5 B)\epsilon_1 - (1 \leftrightarrow 2)$$

Using the spinor technology, the A terms add up, while the B ones cancel:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]A = 2\bar{\epsilon_2}\gamma^\mu\partial_\mu A\epsilon_1 \Rightarrow [\delta_{\epsilon_1}, \delta_{\epsilon_2}] = 2\bar{\epsilon_2}\gamma^\mu\epsilon_1 P_\mu \tag{3.18}$$

(a similar conclusion can be drawn for $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]B$. Now, recall that $\{Q_\alpha, Q_\beta\} = 2(\gamma_\mu C)_{\alpha\beta}P^\mu$, and $\delta_\epsilon = \bar{\epsilon}Q$. Hence,

$$\begin{aligned}[\delta_{\epsilon_1}, \delta_{\epsilon_2}] &= [\bar{\epsilon_1}Q, \bar{\epsilon_2}Q] = \bar{\epsilon_1}^\alpha Q_\alpha \bar{\epsilon_2}Q_\beta - \bar{\epsilon_2}^\beta Q_\beta \bar{\epsilon_1}^\alpha Q_\alpha = -\bar{\epsilon_1}^\alpha \bar{\epsilon_2}Q_\alpha Q_\beta - \bar{\epsilon_1}^\alpha \bar{\epsilon_2}^\beta Q_\beta Q_\alpha \\ &= -\bar{\epsilon_1}^\alpha \bar{\epsilon_2}^\beta 2(\gamma_\mu C)_{\alpha\beta}P^\mu = -2\bar{\epsilon_1}^\alpha (\gamma_\mu)_\alpha{}^\gamma \epsilon_{2\gamma}P^\mu = -2\bar{\epsilon_1}\gamma_\mu\epsilon_2 P^\mu = 2\bar{\epsilon_2}\gamma^\mu\epsilon_1 P_\mu\end{aligned}$$

Which is in agreement with the previous result! Now, what about χ?

$$\begin{aligned}[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\chi &= \delta_{\epsilon_1}[\partial\!\!\!/(A + i\gamma_5 B)\epsilon_2] - (1 \leftrightarrow 2) = \delta_{\epsilon_1}[\partial_\mu A\gamma^\mu\epsilon_2 + i\gamma^\mu\gamma_5\partial_\mu B\epsilon_2] - (1 \leftrightarrow 2) \\ &= \partial_\mu(\bar{\epsilon_1}\chi)\gamma^\mu\epsilon_2 + i\gamma^\mu\gamma_5\partial_\mu(\bar{\epsilon_1}i\gamma_5\chi)\epsilon_2 - (1 \leftrightarrow 2) \\ &= (\bar{\epsilon_1}\partial_\mu\chi)\gamma^\mu\epsilon_2 - (\bar{\epsilon_1}\gamma_5\partial_\mu\chi)\gamma^\mu\gamma_5\epsilon_2 - (1 \leftrightarrow 2)\end{aligned}$$

It seems that this is hopeless, we have reached a dead end. However, we can use the so-called Fierz identities (only for Majorana spinors):

$$(\bar{\lambda}M\chi)N\psi = \frac{1}{4}\sum_I(\bar{\lambda}\vartheta^I\psi)(N\vartheta_I M\chi) \tag{3.19}$$

where M, N are matrices, λ, χ and ψ are spinors, and

$$\begin{cases}\vartheta^I &= \{I, \gamma^\mu, i\gamma^\mu\gamma^5, \gamma^5, i\gamma^{\mu\nu}\} \\ \vartheta_I &= \{I, \gamma_\mu, i\gamma_\mu\gamma_5, \gamma_5, i\gamma_{\mu\nu}\}\end{cases}$$

(here we have taken $\gamma^{\mu\nu} \equiv \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}$, with $\mu < \nu$). For completeness, $\mathrm{Tr}\,\vartheta^I\vartheta_I = 4\delta^i_J$.

<u>Claim:</u> Only the terms with $\vartheta_I = \{\mu, i\gamma_{\mu\nu}\}$ contribute to $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\chi$. Furthermore, when plugged in, the $i\gamma_{\mu\nu}$ terms cancel (see 4.4, Problem 3).

We are thus left with

$$\begin{aligned}[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\chi &= -\frac{1}{2}(\bar{\epsilon_1}\gamma^\rho\epsilon_2)\gamma^\mu\gamma_\rho\partial_\mu\chi - (1 \leftrightarrow 2) \\ &= -\frac{1}{2}(\bar{\epsilon_1}\gamma^\rho\epsilon_2)(-\gamma_\rho\gamma^\mu + 2\delta^\mu_\rho)\partial_\mu\chi - (1 \leftrightarrow 2) \\ &= \frac{1}{2}(\bar{\epsilon_1}\gamma^\rho\epsilon_2\gamma_\rho)\partial\!\!\!/\chi - \bar{\epsilon_1}\epsilon_2\partial\!\!\!/\chi - (1 \leftrightarrow 2)\end{aligned}$$

We can clearly see that the algebra on χ does not close unless we impose the equation of motion $\not{\partial}\chi = 0$. Therefore the SUSY algebra only closes on-shell! This means that the transformations we wrote are not a "faithful" representation of the SUSY algebra off-shell, even if the action is invariant. But why did this happen?

The fact that SUSY doesn't close off-shell is not a problem *per se*, in fact it was expected as the degrees of freedom in Table 3.2 didn't add up off-shell. If we want it to close off-shell too (because for example in QFT we need it as we integrate over all possible field configurations, including off-shell), we need to add the previously mentioned auxiliary fields F, G, with their own supersymmetric transformations.

Invariance of the theory & interactions

Let us now show that the Lagrangian is indeed invariant:

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{WZ}} &= -(\partial_\mu A)\partial^\mu \delta A - (\partial_\mu B)\partial^\mu \delta B - \bar{\chi}\not{\partial}\delta\chi = \partial_\mu\partial^\mu A\delta A + \partial_\mu\partial^\mu B\delta B - \bar{\chi}\not{\partial}\delta\chi \\ &= \Box A\bar{\epsilon}\chi + \Box B\bar{\epsilon}i\gamma_5\chi - \bar{\chi}\not{\partial}\left[\not{\partial}(A + i\gamma_5 B)\epsilon\right]\end{aligned}$$

where in the first line we dismissed boundary terms. If we use the spinor technology, concretely we will need

$$\not{\partial}\not{\partial} = \gamma^\mu\partial_\mu\gamma^\nu\partial_\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_\mu\partial^\mu \tag{3.20}$$

we then plug it in the variation, and it turns out to be zero, $\delta\mathcal{L}_{WZ} = 0$, as it should. This is our first example of a SUSY action! Supersymmetry was a hidden symmetry of a very simple action that we have seen many times.

The action can be generalized to include masses and interactions:

$$\mathcal{L}_{\mathrm{WZ}} = -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}m^2(A^2 + B^2) - \frac{1}{2}\bar{\chi}(\not{\partial} - m)\chi \tag{3.21}$$

$$- g\bar{\chi}(A + i\gamma_5 B)\chi - mg(A^3 + AB^2) - \frac{g^2}{2}(A^2 + B^2)^2 \tag{3.22}$$

This is the most general interactive, renormalizable, supersymmetric action. Note however that not all coefficients are independent, they must be related (this is an example of the power of supersymmetry). Also, the existence of bosonic global symmetries implies the existence of a conserved current (which we will not derive here),

$$J^\mu \equiv \not{\partial}(A - i\gamma_5 B)\gamma^\mu\chi \tag{3.23}$$

The same is true for the global fermionic currents. These "supercurrents" will sit in supersymmetry multiplets.

Lecture 4

$\mathcal{N} = 4$ Super Yang-Mills & 't Hooft's large N limit

After studying the Wess-Zumino model, and before diving into the realizations of the extended SUSY algebra, we will consider the vector multiplet (λ_α, A_μ). Not only it is interesting on its own, but it will also be instructive for the later generalizations.

4.1 $\mathcal{N} = 1$ Super QED and Yang-Mills

Let us study the abelian and non-abelian cases for A_μ separately. Here we will be interested in the free theories, although generalizations are possible.

- Abelian: The natural action for the fields is given by the following Lagrangian:

$$\mathcal{L}_{\text{SYM}} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \bar{\lambda} \not{\partial} \lambda + \tag{4.1}$$

  The SUSY transformations are given now by

$$\begin{cases} \delta A_\mu & = \bar{\epsilon} \gamma_\mu \lambda \\ \delta \lambda & = -\frac{1}{4} F_{\mu\nu} \gamma^{\mu\nu} \epsilon \end{cases} \tag{4.2}$$

  These transformation close on-shell. One can check that $\delta \mathcal{L} = \partial_\mu K^\mu$, provided the Bianchi identities $dF = 0 \Rightarrow \partial_{[\rho} F_{\mu\nu]} = 0$. There is no interaction $(\bar{\lambda} \not{D} \lambda)$, as had λ been charged under $U(1)$, that would destroy its Majorana condition. The same goes for A_μ.

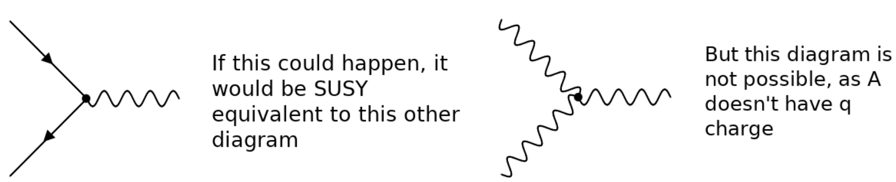


Figure 4.1: Example of SUSY vertex constrains. This is also consistent with the above discussion on the Majorana condition, which does not allow for this vertex to happen.

- Non-abelian: The generalization is straightforward, the Lagrangian description (for $\lambda_\alpha^a, A_\mu^a; \quad a = 1 ... \dim G$) in this case is given by

$$\mathcal{L}_{\text{SYM}} = \text{Tr}\left[-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\bar{\lambda}\not{D}\lambda\right] \tag{4.3}$$

  Now the fields are charged under G (for example in usual QCD the gluons are charged with color charge). All fields in a SUSY multiplet are in the same representation of the gauge group ($[Q, T_G] = 0$, by Coleman-Mandula). In particular, all fields in the vector multiplet are in the adjoint representation.

  Another important property is that there is an additional classical symmetry of the action: scale invariance

$$x \to \Omega^1 x, \quad A_\mu \to \Omega^{-1}A_\mu, \quad \lambda \to \Omega^{-3/2}\lambda \tag{4.4}$$

  (where the powers of Ω are the classical dimensions of the fields). However at the quantum level this symmetry is broken, unless the beta function vanishes (we will see this come back in (4.18)).

The above discussion refers to Super Yang-Mills models, however we can also consider Super QED, which involves adding matter multiplets (chiral ones, with (ϕ, χ_α), with $\phi = A + iB$ a complex scalar) to the Lagrangian.

4.2 Irreducible representations of SUSY (extended)

One may consider massive/massless representations of supersymmetry. As an example, say we want to find representations of SO(3,1), that is, the Lorentz group. It's non-compact, so it has infinite dimensional representations. That is bothersome, so we use the method of induced representations, by Wigner (we find a representation of a compact subgroup leaving p_μ invariant, and extend it to the full group):

Particles can be massive, therefore having SO(3) as their Little group (as we can write $p^\mu = (m, 0, 0, 0)$ in the rest frame, $p^2 = -m^2$), or massless, having SO(2) as their Little group (and therefore having momenta given by $p^\mu = (k, k, 0, 0)$, $p^2 = 0$).

We now have to do the same for (extended) SUSY: $\{Q_\alpha^i, Q_\beta^j\} = 2(C\gamma_\mu)_{\alpha\beta}p^\mu\delta^{ij}$.

Take a massless particle with momentum $q_\mu = (k, 0, 0, k)$. It can be represented by the state $|q, j\rangle \to P^\mu |q, j\rangle = q^\mu |q, j\rangle$. This means

$$\{Q_\alpha^i, Q_\beta^j\} |q\rangle = 2\delta^{ij}(\gamma_\mu C)_{\alpha\beta}q^\mu \tag{4.5}$$

For convenience, $\underbrace{\alpha}_{\text{SU(3,1)}} \to \underbrace{A, \dot{A}}_{\text{SU(2)}\times\text{SU(2)}}$; $A, \dot{A} = \{1, 2\}$. With this,

$$\{Q^{Ai}, Q_j^{\dot{B}}\} = -2\delta_j^i(\sigma_0 + \sigma_3)^{A\dot{B}}k = 4k\delta_j^i\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}^{A\dot{B}}$$

Where the acting on $|q\rangle$ is implicit, and $\sigma^\mu = (\mathbb{I}, \sigma^1, \sigma^2, \sigma^3)\,, \sigma_\mu = (-\mathbb{I}, \sigma^1, \sigma^2, \sigma^3)$. This entails that the only commutator which acts non-trivially on the state is

$$\{Q^{2i}, Q_j^{\dot{2}}\} = 4k\delta_j^i \tag{4.6}$$

The triviality of $\{Q^{1i}, Q^{\dot{1}}_j\} = 0$ along with the relation $(Q^{\dot{1}}_1)^* = -Q^{1i}$ imply

$$\langle q|Q^{1i}(Q^{1i})^* + (Q^{1i})^* Q^{1i}|q\rangle = 0 \tag{4.7}$$

If $|q\rangle$ is positive definite (which always happens in QFT), then

$$Q^{1i}\,|q\rangle = Q^{\dot{1}}_i\,|q\rangle = 0; \quad Q^i_2\,|q\rangle = Q_{\dot{2}i}\,|q\rangle = 0 \tag{4.8}$$

From now on, for the sake of notation, we will rename the Latin indices as $i \equiv I, j \equiv J$, and furthermore, we will denote $Q^I_1 \equiv \mathcal{Q}^I$ Let's look at other commutation relations for Q's:

$$[Q^I_A, M_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\nu})^B_A Q^I_B \Rightarrow [\mathcal{Q}^I, M_{12}] = -\frac{i}{2}\mathcal{Q}^I; \quad [(\mathcal{Q}^I)^*, M_{12}] = \frac{i}{2}(\mathcal{Q}^I)^* \tag{4.9}$$

The operators $\mathcal{Q}^I, (\mathcal{Q}^I)^*$ behave as raising and lowering operators! More explicitly: assume a state with helicity λ, then $J\,|\lambda\rangle = i\lambda\,|\lambda\rangle$. Acting with $\mathcal{Q}^I$,

$$J\mathcal{Q}^I\,|\lambda\rangle = \big([J, \mathcal{Q}^I] + \mathcal{Q}^I J\big)\,|\lambda\rangle = \left(\frac{i}{2}\mathcal{Q}^I + \mathcal{Q}^I i\lambda\right)|\lambda\rangle = \mathcal{Q}^I i\left(\lambda + \frac{1}{2}\right)|\lambda\rangle$$

This shows that $\mathcal{Q}^I$ is a raising operator $\lambda \to \lambda + 1/2$. Analogously, $(\mathcal{Q}^I)^*$ is a lowering operator $\lambda \to \lambda - 1/2$.

What is the implication of this? We can now build representations in the usual fashion: define a vacuum $|\lambda_{max}\rangle$ such that $\mathcal{Q}^I\,|\lambda_{max}\rangle$. From this state, we can build a whole tower of states by acting with the lowering operator on them. By the nilpotency $((\mathcal{Q}^I)^*)^2 = 0$, the tower has the form

$$|\lambda_{max}\rangle \to (\mathcal{Q}^I)^*\,|\lambda_{max}\rangle \to (\mathcal{Q}^I)^*(\mathcal{Q}^J)^*\,|\lambda_{max}\rangle \to \ldots \tag{4.10}$$

Example (4d, $\mathcal{N} = 1 \Rightarrow I = J = 1$, $\lambda_{max} = 1/2$): The tower of states obtained is the following

$$|1/2\rangle \to (\mathcal{Q})^*\,|1/2\rangle \to (\mathcal{Q}^2)^*\,|1/2\rangle = 0$$

You might not notice it, but we have rediscovered the Chiral multiplet! The first and second states (the non-zero ones) are, respectively, χ_α ($\lambda = 1/2$) and A ($\lambda = 0$), along with their CPT conjugates (forced by the CPT invariance of quantum field theory) χ_α ($\lambda = -1/2$) and B ($\lambda = 0$).

$$(1/2, 0) \oplus (0, -1/2) \Rightarrow (\chi_\alpha, A, B) \tag{4.11}$$

Example (4d, $\mathcal{N} = 1 \Rightarrow I = J = 1$, $\lambda_{max} = 1$): In this case, if we follow the same procedure we end up with the Vector multiplet:

$$(1, 1/2) \oplus (-1/2, -1) \Rightarrow (\chi_\alpha, A_\mu) \tag{4.12}$$

If we began with $\lambda_{max} = 3/2$, we would end up with the so-called Current supermultiplet (j^α_μ, j_μ). This is not an elementary multiplet, though.

Let us now work out in detail the 4d $\mathcal{N} = 4$, λ_{max} case, as it will be relevant in what follows. There are four supercharges $\mathcal{Q}^I$, $\quad I = 1, 2, 3, 4$. We can build the tower of states 4.1

States	# of states	$\times$	Helicity
$\|1\rangle$	$\binom{4}{0}=1$	$\times$	1
$(\mathcal{Q}^1)^*\|1\rangle, \quad (\mathcal{Q}^2)^*\|1\rangle, \quad (\mathcal{Q}^3)^*\|1\rangle, \quad (\mathcal{Q}^4)^*\|1\rangle$	$\binom{4}{1}=4$	$\times$	1/2
$(\mathcal{Q}^1)^*(\mathcal{Q}^2)^*\|1\rangle, \quad (\mathcal{Q}^1)^*(\mathcal{Q}^3)^*\|1\rangle, \quad (\mathcal{Q}^1)^*(\mathcal{Q}^4)^*\|1\rangle, \dots$	$\binom{4}{2}=6$	$\times$	0
$(\mathcal{Q}^1)^*(\mathcal{Q}^2)^*(\mathcal{Q}^3)^*\|1\rangle, \quad (\mathcal{Q}^1)^*(\mathcal{Q}^2)^*(\mathcal{Q}^4)^*\|1\rangle, \dots$	$\binom{4}{3}=4$	$\times$	-1/2
$(\mathcal{Q}^1)^*(\mathcal{Q}^2)^*(\mathcal{Q}^3)^*(\mathcal{Q}^4)^*\|1\rangle$	$\binom{4}{4}=1$	$\times$	-1

Table 4.1: $\mathcal{N}=4$ tower of states

We therefore have discovered the $\mathcal{N}=4$ Vector multiplet $(A_\mu, \chi^I_\alpha, \Phi_{IJ})$, where Φ_{IJ} is antisymmetric. This multiplet, being self-conjugate, doesn't need the CPT conjugate we had to add in the previous examples.

$\mathcal{N}=4$ is called a "maximally supersymmetric theory" (with spin $s \leq 1$). If we had more supercharges, and began with $|1\rangle$, we would have gone further than $|-1\rangle$. A nice way to represent the multiplet is through what is usually called the "rhombic diagram", which also allows us to identify submultiplets (Figure Figure 4.2)

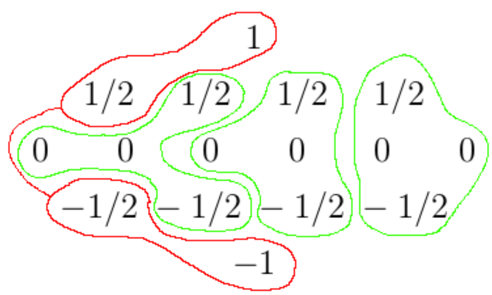


Figure 4.2: Rhombic diagram of the multiplet, with submultiplets indicated

This means that

$$(\mathcal{N}=4 \quad \text{Vector}) = (\mathcal{N}=1 \quad \text{Vector}) + 3 \otimes (\mathcal{N}=1 \quad \text{Chiral})$$

Another way to classify the states is through the way R-symmetry acts on them. Remember that $[Q^I_\alpha, R] = U^I{}_J Q^I_\alpha$, then $U^\dagger = -U$. This means that R-symmetry is a $\mathrm{U}(\mathcal{N})_R$ symmetry.

The $\mathcal{N}=4$ Vector multiplet $(A_\mu, \chi^I_\alpha, \Phi_{IJ})$ then transforms in the $(1, 4, 6)$ representations of $\mathrm{U}(4)_R$

4.2.1 The action of $\mathcal{N}=2,3$ Super Yang-Mills

Although we will be mostly interested in the case featured in the next subsection, we will also briefly describe the $\mathcal{N}=2$ model. Its field content is $\mathcal{N}=2$ vector multiplet: two scalar fields G, H, a vector field A_μ and a Majorana fermion, the spin-1/2 field χ_I (if we want the algebra to close off-shell we would also need to include three bosonic degrees of freedom $X^{ij} = X^{ji}$). The invariant action (for the free theory) can be built as

$$\mathcal{L}_{\text{SYM}} = \text{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu G)(\partial^\mu G) - \frac{1}{2}(\partial_\mu H)(\partial^\mu H) - \frac{1}{2}\bar{\chi}^I \not{\partial}\chi_I\right) \tag{4.13}$$

This model does enjoy the supersymmetric transformation

$$\begin{cases} \delta G &= i\bar{\epsilon}^I \chi_I \\ \delta H &= \bar{\epsilon}^I \gamma_5 \chi_I \\ \delta A_\mu &= \bar{\epsilon}^I \gamma_\mu \chi_I \\ \delta \chi_I &= -\frac{1}{2}(\sigma_{\mu\nu})F^{\mu\nu}\epsilon_I - i\not{\partial}(F - i\gamma_5 G)\epsilon_I \end{cases} \tag{4.14}$$

And their respective equations of motion are given by

$$\partial_\mu \partial^\mu G = \partial_\mu \partial^\mu H = \not{\partial}\chi_I = \partial_\mu F^{\mu\nu} = 0 \tag{4.15}$$

As in the $\mathcal{N} = 1$ case, we can add a hypermultiplet in order to have $\mathcal{N} = 2$ SuperQED.

The case for $\mathcal{N} = 3$: This case is usually not discussed, as if one constructs its multiplets, and adds their CPT conjugates the resulting content is the same as the $\mathcal{N} = 4$ one.

4.2.2 The (unique) action of $\mathcal{N} = 4$ Super Yang-Mills

The Lagrangian corresponding to the $\mathcal{N} = 4$ Vector multiplet is the so-called $\mathcal{N} = 4$ Super Yang-Mills (SYM) model:

$$\begin{aligned} \mathcal{L}_{\text{SYM}} = \text{Tr}\Bigg(&-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu \Phi_{IJ} D^\mu \Phi^{IJ} - \bar{\lambda}^I \not{D}\lambda \\ &+ i\bar{\lambda}^I[\lambda^J, \Phi_{IJ}] + \text{h.c.} - \frac{1}{4}[\Phi_{IJ}, \Phi_{KL}][\Phi^{IJ}, \Phi^{KL}]\Bigg), \end{aligned} \tag{4.16}$$

where $I, J = 1, ..., 4$ and Φ_{IJ} is antisymmetric, therefore it constitutes 6 scalar degrees of freedom (the ones present in Figure 4.2). The form of the Lagrangian and the couplings are uniquely fixed by supersymmetry. This action enjoys the supersymmetric transformations:

$$\begin{cases} \delta A_\mu &= i\bar{\epsilon}^I \gamma_\mu \lambda_I + \text{h.c.} \\ \delta \Phi_{IJ} &= \bar{\epsilon}_I \lambda_J - \bar{\epsilon}_J \lambda_I + \varepsilon_{IJKL}\bar{\epsilon}^K \lambda^L \\ \delta \lambda_I &= \frac{i}{2}(\sigma_{\mu\nu})F^{\mu\nu}\epsilon_I + 2\gamma_\mu \epsilon^J D^\mu \Phi_{IJ} + 2i[\Phi_{IJ}, \Phi^{JK}]\epsilon_K \end{cases} \tag{4.17}$$

where ε is the Levi-Civita symbol (not to be confused with the gauge parameter ϵ). Note that, unlike in the $N = 1, 2$ cases, this action is unique: any attempt to include more multiplets is not allowed, unless we add $\mathcal{L}_{\text{YM}}$ itself. But of course that is trivial.

Symmetries of $\mathcal{N}=4$ Super Yang-Mills

So far we have the following symmetries: Poincaré, Maximal SUSY, and SU(4) R-symmetry[1]. There is one more we shall study: conformal symmetry.

Beta functions in Yang-Mills theories Consider a general QFT with U(N) gauge group (A_μ), n_f number of Majorana fermions and n_s number of real scalars, both in the adjoint representation. Let's look at $\lambda_I = g^2_{\text{YM}}$, for this model. An interaction vertex is $g_{\text{YM}}\bar{\lambda}\not{A}\lambda \subset \mathcal{L}$, studying what happens with it after quantum corrections, we find

$$\beta_{\text{YM}} = -N\frac{g^2_{\text{YM}}}{48\pi^2}\left(11 - 2n_f - \frac{1}{2}n_s\right) \tag{4.18}$$

If we compare this to $\mathcal{N}=4$ SYM, which has $n_s = 6$, $n_f = 4$, we find that $\beta_{\text{SYM}} = 0$!.
Conformal symmetry from the vanishing of beta functions Qualitatively the picture is quite clear: a vanishing beta function indicates a fixed point in the theory, where there are no scales, hence it is conformal invariant.

Thus, joining together all the symmetries, this model will enjoy

$$(\text{SO}(4,2) \oplus \max \text{SUSY } \mathcal{N}=4) \in \text{PSU}(2,2|4) \tag{4.19}$$

This supergroup PSU(2,2|4) (where PSU stands for Projective Special Unitary group) contains as a subgroup SO(4,2)× SO$(6)_R$, which has the same symmetry as AdS$_5 \times S^5$, respectively... this already smells like the duality.

In String Theory, the way to "engineer" field theories is by the use of branes. Consider D3-branes (3+1 dimensional QFTs, therefore with SU(4)) in 10d. The Lorentz group in 10 dimensions SO(9,1) will be broken by the introduction of these branes into SO(3,1)× SO(6) (corresponding to the worldvolume Lorentz group, and the 6 orthogonal dimensions to the branes). If we N_c of them, the gauge group of the strings on them will be SU(N_c).

Let us now set this aside for a while, as there is something we have to present before.

4.3 't Hooft's large N limit

In gauge theories, the standard limit is the perturbative one, $g \to 0$, which is what is used in the Feynman diagrams approach, neglecting higher order terms. The problem is that it is limited to perturbations around the classical theory.

The 't Hooft's large N limit is an interesting non-trivial limit of QFTs (not necessarily supersymmetric), which can be applied to: O(N) models (N scalars) where under the large N limit it becomes exactly solvable; or gauge theories, with large rank$(G) = N-1$, with for instance $G = $ SU(N).

[1] It would be tempting to think that, given that there are four supercharges Q^I_α, the R-symmetry rotating them constitutes an U(4) group. That is indeed true, however it is not hard to see that in general U(N) =U(1)×SU(N), as one can always factor out an $e^{i\alpha}$, $\alpha \in \mathbb{R}$ from the defining U(N) matrix, while the remaining matrix belongs to SU(N). In the case at hand, the U(1) part of the R-symmetry is broken in the Lagrangian, with only the residual SU(4) symmetry remaining.

’t Hooft’s limit is not an expansion around a classical theory, in fact it is the opposite in general, a very strongly coupled quantum system. Through holography, it is perturbative around classical gravity in $d{+}1$ dimensions.

It all begins with a basic observation. Suppose you have a gauge theory, with Lagrangian

$$\mathcal{L} = \mathrm{Tr}\left(-\frac{1}{4g_{YM}}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\bar{q}\not{D}q - \frac{1}{2}\bar{\chi}\not{D}\chi\right) \tag{4.20}$$

(where q and χ could be generic quarks and gauginos). The “effective” coupling is given by $\lambda = g_{YM}^2 N$. To see this, consider

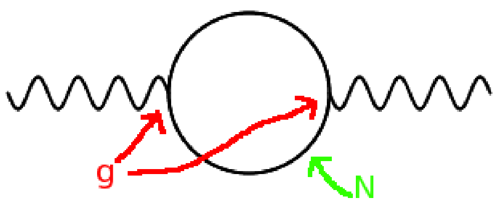
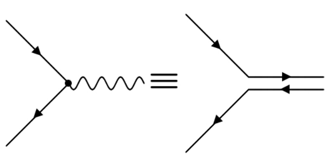


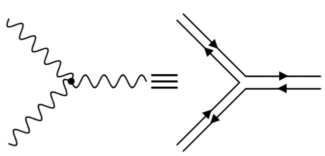
Figure 4.3: Here N is the number of loops, and as it can be seen there are two coupling constants g_{YM} contributing to each order.

In this setting, ’t Hooft’s limit is $N \to \infty$, but keeping λ fixed (hence letting $g^2 \to 0$).

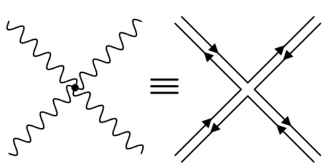
Double line notation: Suppose we have a theory containing quarks, where the interactions vertices are, for example, $\bar{q}^i (A_\mu)_i^j q_j$. The quarks q_j are represented with oriented lines (arrows), while the gauge fields $(A_\mu)_j^i$ instead of being represented with a wiggly line, they will be represented with two lines of opposite directions (one for the “quark index” j and the other for the “antiquark index” i). When we contract the indices with their corresponding quark/antiquark, we glue lines. Let us see this with some examples of interactions:



- Quark-antiquark interaction: $\bar{q}^i (A_\mu)_i^j q_j$



- Triple gauge field interaction: $A^2 \partial A$



- Quadruple gauge field interaction A^4

Now, consider only fields in the adjoint representation, and vacuum diagrams, i.e. no external lines (Figure 4.4)

These diagrams translate to the followings with the double line notation (Figure 4.5)

The first two diagrams are planar diagrams, that meaning that they can be drawn on a plane or a sphere, however the third one is not! There is a crossing, which cannot be implemented in a flat surface accurately.

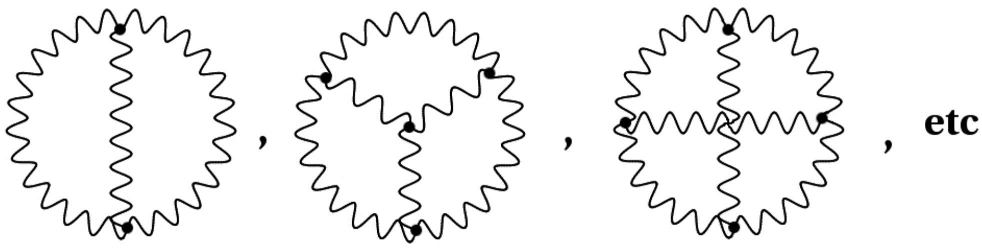


Figure 4.4

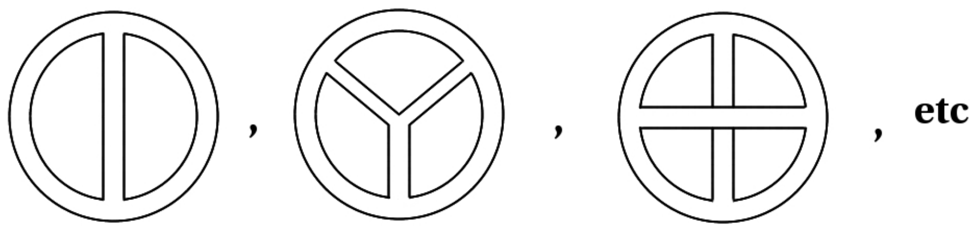


Figure 4.5

Lets look at how these diagrams scale with N and g^2: If we name the propagators by E, the interaction vertices by V and the loops by L, then a diagram with (V, E, L) has

$$\left(\frac{N}{\lambda}\right)^V \left(\frac{\lambda}{N}\right)^E N^L = N^{V-E+L}\lambda^{E-V} = N^{\chi}\lambda^{E-V} \tag{4.21}$$

where χ is the Euler characteristic of the diagram.

Theorem: $\chi = V - E + L = 2 - 2g$, where g is the genus of the diagram (the number of "holes").

Therefore we have for any diagram $N^{2-2g}\lambda^{E-V}$. For example, with the three previous diagrams, we obtain

$$\begin{array}{ccc} (2,3,3) & (4,6,4) & (4,6,2) \\ \lambda N^2 & \lambda N^2 & \lambda^2 N^0 \end{array}$$

For large N (keeping λ finite), the terms that are dominant will be the ones with genus zero, exactly the planar ones! If we compute vacuum amplitudes, we obtain

$$\langle \text{vacuum}|\text{vacuum}\rangle = \mathcal{Z}_{\text{gauge}} = \sum_{g,p} N^{2-2g}\lambda^p C_{g,p} \tag{4.22}$$

The sum over the genus can be understood as a sum over topologies... And here is where string theory makes its appearance, as in string interactions there is also a sum over g_s, the genus of the worldsheet.

Notice that this is then a very quantum limit, unlike the perturbative one, as the sum over topologies will take into account the planar graphs with many loops. In fact, $N \to \infty$, $\lambda \to \infty$, is a

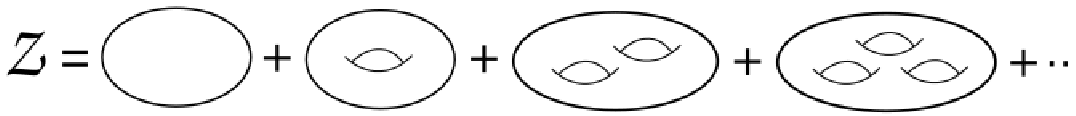


Figure 4.6: This observation in gauge theories led Gerard 't Hooft to conjecture the existence of a string theory lurking this area, one of the first clues towards the AdS/CFT conjecture.

strongly-coupled-highly-quantum limit of the theory. What 't Hooft proposed was an equivalence between the string partition function and the gauge partition function:

$$\mathcal{Z}^{4d}_{\text{gauge}} = \mathcal{Z}^{?d}_{\text{string}} \tag{4.23}$$

The unknown dimension of the string theory turned out to be $? = 10$. Juan Maldacena's proposal of AdS/CFT makes this equivalence more precise, as

$$\mathcal{Z}_{\mathcal{N}=4\,\text{SYM}} = \mathcal{Z}^{AdS_5\times S^5}_{\text{string}} \tag{4.24}$$

This is believed to be true for any λ or N. For the case of large N and large λ limits of $\mathcal{N}=4$ SYM, it is even more concrete, as it it equivalent to type IIB superstrings on $AdS_5 \times S^5$.

4.4 Exercises (Lectures 3 & 4)

Problem 1: SUSY algebra

a) Consider a set of bosonic generators $B_{1,2,3}$ and Fermionic generators $F_{1,2,3}$. Prove the following super-Jacobi identities by explicit computation:

$$[[B_1, B_2], B_3] + [[B_3, B_1], B_2] + [[B_2, B_3], B_1] = 0\,, \tag{4.25}$$

$$[[B_1, B_2], F_3] + [[F_3, B_1], B_2] + [[B_2, F_3], B_1] = 0\,, \tag{4.26}$$

$$\{[B_1, F_2], F_3\} - \{[F_3, B_1], F_2\} + [\{F_2, F_3\}, B_1] = 0\,, \tag{4.27}$$

$$[\{F_1, F_2\}, F_3] + [\{F_3, F_1\}, F_2] + [\{F_2, F_3\}, F_1] = 0\,, \tag{4.28}$$

where we have defined the commutator and anti-commutator,

$$[X, Y] \equiv XY - YX\,, \qquad \{X, Y\} \equiv XY + YX\,. \tag{4.29}$$

b) Consider the following Ansatz for the commutation relation of the supercharge with the translation operator:

$$[Q_\alpha, P_\mu] = c\,(\gamma_\mu)_\alpha^{\ \beta}\, Q_\beta\,, \tag{4.30}$$

where c is a real constant and γ_μ satisfy the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. Using the super-Jacobi identity for (Q_α, P_μ, P_ν), prove that $c = 0$.

Comment: The most general Ansatz one can consider is

$$[Q_\alpha, P_\mu] = (c\gamma_\mu + d\gamma_\mu\gamma_5)_\alpha^{\ \beta}\, Q_\beta\,. \tag{4.31}$$

One can similarly show that $c = d = 0$, but you are not asked to do this.

Problem 2: Majorana Spinors

Assume ϵ, χ are Majorana spinors, i.e.,

$$\epsilon_\alpha = C_{\alpha\beta}\bar{\epsilon}^\beta\,, \qquad \chi_\alpha = C_{\alpha\beta}\bar{\chi}^\beta\,, \tag{4.32}$$

where C is the charge-conjugation matrix, whose defining property is

$$C\gamma^\mu C^{-1} = -\left(\gamma^\mu\right)^T\,, \qquad C^T = -C\,, \tag{4.33}$$

and satisfies

$$C_{\alpha\beta} = -C_{\beta\alpha}\,, \qquad C_{\alpha\beta}C^{\gamma\beta} = -C_{\alpha\beta}C^{\beta\gamma} = -\delta_\alpha^\gamma\,. \tag{4.34}$$

Given a matrix $M_\alpha^{\ \beta}$ and two Majorana spinors ϵ, χ, we construct bilinear as follows

$$\bar{\epsilon}M\chi \equiv \bar{\epsilon}^\alpha M_\alpha^{\ \beta}\chi_\beta\,. \tag{4.35}$$

a) Using the properties above, show the following identity used in class:

$$\bar{\epsilon}\chi = \bar{\chi}\epsilon\,. \tag{4.36}$$

Hint: Recall that each component of a spinor is an anticommuting variable, e.g.,

$$\bar{\chi}^\beta\bar{\epsilon}^\alpha = -\bar{\epsilon}^\alpha\bar{\chi}^\beta\,, \qquad \epsilon_\alpha\chi_\beta = -\chi_\beta\epsilon_\alpha\,, \qquad \epsilon_\alpha\bar{\chi}^\beta = -\bar{\chi}^\beta\epsilon_\alpha\,, \qquad \text{etc...}$$

b) Note that it follows from (4.33) that

$$(\gamma^\mu C)^T = \gamma^\mu C \qquad \Rightarrow \qquad (\gamma^\mu)_\alpha^{\ \beta} C_{\beta\delta} = (\gamma^\mu)_\delta^{\ \beta} C_{\beta\alpha} \tag{4.37}$$

Using this property, show that

$$\bar\epsilon\gamma^\mu\chi = -\bar\chi\gamma^\mu\epsilon\,. \tag{4.38}$$

c) Repeating the same logic as in part **a)** and **b)**, show the remaining identities:

$$\bar\epsilon\gamma_5\chi = \bar\chi\gamma_5\epsilon\,, \qquad \bar\epsilon\gamma^\mu\gamma_5\chi = \bar\chi\gamma^\mu\gamma_5\epsilon\,, \qquad \bar\epsilon\gamma_{\mu\nu}\chi = -\bar\chi\gamma_{\mu\nu}\epsilon\,, \tag{4.39}$$

where $\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\gamma_{\mu\nu} \equiv \frac{1}{2}[\gamma_\mu, \gamma_\nu]$.

Hint: To show (4.39) you need to first establish properties similar to (4.37).

Problem 3: Closure of SUSY algebra

We showed in class that the following supersymmetry transformations:

$$\begin{aligned} \delta A &= \bar\epsilon\chi\,, \\ \delta B &= i\bar\epsilon\gamma_5\chi\,, \\ \delta\chi &= \gamma^\mu\partial_\mu(A + i\gamma_5 B)\epsilon\,, \end{aligned}$$

close to the SUSY algebra on the fields A, B. Indeed,

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]A = 2\bar\epsilon_2\gamma^\mu\epsilon_1 P_\mu A\,, \tag{4.40}$$

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]B = 2\bar\epsilon_2\gamma^\mu\epsilon_1 P_\mu B\,, \tag{4.41}$$

as expected from the SUSY algebra. However, this is not the case on χ. Indeed, **complete the steps below** to show that

$$\begin{aligned} [\delta_{\epsilon_1}, \delta_{\epsilon_2}]\chi &= \delta_{\epsilon_1}\delta_{\epsilon_2}\chi - (1 \leftrightarrow 2) \\ &\ \,\vdots \\ &= \underbrace{2\bar\epsilon_2\gamma^\mu\epsilon_1 P_\mu\chi}_{\text{SUSY algebra}} - (\bar\epsilon_2\gamma^\rho\epsilon_1)\,\gamma_\rho \underbrace{\gamma^\mu\partial_\mu\chi}_{\propto\text{ EOM}}\,. \end{aligned}$$

Thus, the SUSY algebra closes only on-shell, i.e., when the equation of motion (EOM) for χ is imposed.

Hint: As discussed in class, you need to use the Fierz identity

$$(\bar\lambda M\chi)\,N\psi = -\frac{1}{4}\sum_I (\bar\lambda\mathcal{O}^I\psi)\,N\mathcal{O}_I M\chi\,, \tag{4.42}$$

where $\mathcal{O}^I = \{I, \gamma^\mu, i\gamma^\mu\gamma^5, \gamma^5, i\gamma^{\mu\nu}\}$, $\mathcal{O}_I = \{I, \gamma_\mu, i\gamma_\mu\gamma_5, \gamma_5, i\gamma_{\mu\nu}\}$, and $\mu < \nu$, to reorder the position of fermions. Then,

- Show that the only terms contributing to $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\chi$ are from $\mathcal{O}^I = \{\gamma^\mu, i\gamma^{\mu\nu}\}$.
- Show that the term with $\mathcal{O}^I = i\gamma^{\mu\nu}$ in fact also cancels, after using $\gamma_5\gamma_\mu\gamma_5 = -\gamma_\mu$ and $\gamma_5\gamma_{\mu\nu}\gamma_5 = \gamma_{\mu\nu}$.

Problem 4: Representation theory

In this problem you are asked to repeat the analysis we did in class (for $\mathcal{N} = 1, 4$) to build massless representations of $\mathcal{N} = 2$ and $\mathcal{N} = 3$ supersymmetry.

We define the vacuum state $|\lambda_{\rm max}\rangle$ as satisfying

$$(\mathcal{Q}^I)^*|\lambda_{\rm max}\rangle = 0\,, \qquad I = 1, \cdots, \mathcal{N}\,.$$

a) There are two basic multiplets in $\mathcal{N} = 2$ theories: the $\mathcal{N} = 2$ *vector multiplet* and the *hypermultiplet*, corresponding to $\lambda_{\rm max} = 1$ and $\lambda_{\rm max} = \frac{1}{2}$, respectively. Construct these two multiplets, showing the entire tower of states, how they are obtained from the vacuum, and their helicity:

state $\vert\lambda\rangle$	helicity λ
$\vert\lambda_{\rm vac}\rangle$	0
$\vdots$	$\vdots$

(Keep in mind that $(\mathcal{Q}^I)^*(\mathcal{Q}^J)^* = -(\mathcal{Q}^J)^*(\mathcal{Q}^I)^*$.)

b) Add the CPT conjugate to the states found in part **a)**, if necessary.

- What is the field content (i.e., number of gauge fields A_μ, Majorana fermions χ_α, and real scalars ϕ) of the $\mathcal{N} = 2$ vector multiplet and hypermultiplet?

$$(\mathcal{N} = 2 \text{ vector}) = \{\cdots, \cdots, \cdots\}\,, \qquad (\mathcal{N} = 2 \text{ hyper}) = \{\cdots, \cdots, \cdots\}$$

- How would you decompose these into the $\mathcal{N} = 1$ multiplets discussed in class?
- What is the R-symmetry group and in what representation do all the fields you find transform?

c) Consider the $\lambda_{\rm max} = 1$ multiplet for $\mathcal{N} = 3$. If necessary, add the CPT conjugate. What do you find? Is it something familiar? What do you conclude from this?

Lecture 5

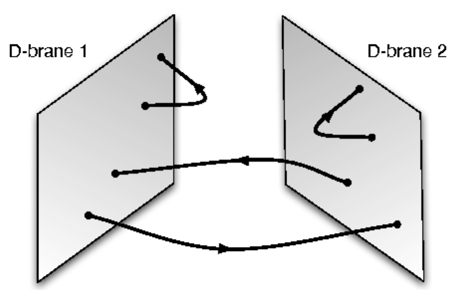
Derivation of the AdS/CFT correspondence from D-branes

Let's recap: We have already studied (Asymptotically) Anti-de Sitter spacetimes, scalar fields living on them, supersymmetric QFT's, and $\mathcal{N} = 4$ Super Yang-Mills. But what does it all have to do with string theory?

It all begins with D-branes. In the previous chapter we rushed through them, as they were not the main topic. We are now going to see them in a bit more detail, as they will be very relevant.

5.1 D-branes

We are going to restrict ourselves to D3-branes, but the properties discussed will be applicable to any Dp-brane. In string theory, if we study open strings, we have to implement boundary conditions at each end, for each of the coordinates of the string. These conditions can be either Neumann or Dirichlet, and the latter ones define "membranes" (hence the name) where the strings can end, hence D-branes.



Say we have a string worldsheet $X^\mu(\sigma,\tau)$, a D3-brane would be the hypersurfaces given by $X^\mu(\sigma = 0, \tau) = X^\mu(\sigma = \pi, \tau) = 0$, $\mu = 0, 1, 2, 3$, while the rest of the components $X^a(\sigma,\tau)$, $a = 4, ..., 9$ would be free (with Neumann conditions).

Computing the spectrum of the string, one finds that it corresponds to gauge fields living on the brane, with gauge group $U(N)$, where N is the number of stacked branes at the endpoint.

In string theory, there are two parameters:

- The string scale, or length: $\alpha' = l_s^2 \sim \dfrac{1}{T}$.
- The string coupling g_s, which features in interactions as it is the genus of the worldsheet.

However in the D-brane context we have seen there is another parameter:

- The number of D-branes stacked N.

Let's introduce N D3-branes in flat space $\mathbb{R}^{1,9}$. In terms of usefulness, there are two possible descriptions, with their properties listed below:

D-brane perspective	Black-brane perspective
Good description if $\lambda = g^2 N \ll 1$ (perturbative)	Opposite case, $\lambda \gg 1$ (non-perturbative)
$g^2 = g_s$	$N = \int_{S^5} \star F^{(5)}$
SU(N) gauge theory	Solution to S_{SUGRA}

Table 5.1: Two different versions of branes in flat space

where

$$S_{\text{SUGRA}} = \frac{1}{(8\pi)^8 (l_p)^8} \int d^{10}x \sqrt{-g} e^{-2\Phi} \left(R + 4(\nabla\Phi)^2 - F^2 \right) \tag{5.1}$$

The black-brane metric is then given by

$$ds^2 = H^{-1/2}(r)\eta_{\mu\nu} dx^\mu dx^\nu + H^{1/2}(r)(dr^2 + r^2 d\Omega_5^2); \quad H(r) = 1 + \frac{\ell^4}{r^4} \tag{5.2}$$

where $\ell^4 = 4\pi g_s N |\alpha'|^2$ (here ℓ has the same role as $2GM$ in Schwarzschild).

When $\lambda \gg 1$, the branes collapse (as a black hole, but with flat event horizon, thus the name black-brane). It is supersymmetric, charged and with zero temperature.

The limits of the black-brane metric are

- $r \to \infty$: Flat space
- $r \to 0$: Horizon
- $r \ll \ell$: Near horizon

We will perform the change of coordinates $z = \ell^2/r$, such that $z \gg \ell$ is the near horizon:

$$ds^2 \approx \frac{\ell^2}{z^2}(\eta_{\mu\nu} dx^\mu dx^\nu) + \frac{\ell^2}{z^2} dz^2 + \ell^2 d\Omega_5^2 = \underbrace{\frac{\ell^2}{z^2}(dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu)}_{\text{AdS}_5} + \ell^2 \underbrace{d\Omega_5^2}_{S^5} \tag{5.3}$$

5.1.1 "The Maldacena limit"

If we leave N, g_s fixed, but take the low energy limit $\alpha' \to 0$, then the will have the two following pictures depending on the brane perspectives:

D-brane picture: There will be massless modes (seen from far away). We can separate the action into

$$S_{\text{tot}} = S_{\text{brane}} + S_{\text{interior}} + S_{\text{faraway}} \tag{5.4}$$

- S_{faraway} will be SUGRA on flat space (far away we don't see branes, only gravitons, which are closed string modes).
- As $S_{\text{interior}} \propto \alpha'$, then $S_{\text{interior}} \to 0$.
- $S_{\text{brane}} = -T_{D3} \int d^4\sigma \sqrt{-\det(P[g_{ab}] + \mathcal{F}_{ab})}$ + fermions, which is the so-called DBI (Dirac-Born-Infeld) action.

We will focus on the last one, the DBI action. The parameters involved in it are

- The brane tension $T_{D3} = [g_s(2\pi)^3\alpha']^{-1}$.
- The pullback of the metric $P[g_{ab}] = \partial_a X^M \partial_b X^N g_{MN}$.
- The total field strength $\mathcal{F}_{ab} = 2\pi\alpha' F_{ab} + B_{ab}$ (where B_{ab} is the so-called Kalb-Ramond field).

The fermion part is also necessary, as we are working in the framework of SUSY.
What is the limit $\alpha' \to 0$ in the brane action? It corresponds to

$$g_{MN} \to \mathbb{R}^{1,9}, \quad \sigma^\mu = X^\mu, \quad X^a = 0, \quad P[g_{ab}] = \delta_{ab}, \quad B_{ab} = 0$$

The brane action is therefore

$$S_{brane} = -T_{D3} \int d^4x \sqrt{-\det(\delta_{ab} + 2\pi\alpha' F_{ab})} \tag{5.5}$$

From linear algebra $\sqrt{\det(\mathbb{I} + \epsilon M)} = \mathbb{I} + \frac{1}{2}\epsilon(\operatorname{Tr} M) + \epsilon^2\left[\frac{1}{2}(\operatorname{Tr} M)^2 - \frac{1}{4}(\operatorname{Tr} M^2)\right]$. The first term is the trivial one, and $\operatorname{Tr} F = 0$, therefore

$$S_{\text{brane}} = -T_{D3}(\alpha')^2 4\pi^2 \int d^4x \frac{1}{4}\operatorname{Tr} F^2 = \frac{1}{2\pi g^2}\int d^4x \frac{1}{4}\operatorname{Tr} F^2 \tag{5.6}$$

This is the usual Yang-Mills action! If we had included the fermionic terms too, this would be the action for Super Yang-Mills theory.

<u>Conclusion:</u> From the D-brane perspective, taking the limit $\alpha' \to 0$ is equivalent to having SUGRA in $\mathbb{R}^{1,9}$ + $\mathcal{N} = 4$ SYM on $\mathbb{R}^{1,3}$.

Black-brane picture: Consider the black-brane metric, with the conformal factor already pulled in front

$$ds^2 = \frac{\ell^2}{z^2}\left[\tilde{H}^{-1/2}(z)\eta_{\mu\nu}dx^\mu dx^\nu + \tilde{H}^{1/2}(z)(dz^2 + z^2 d\Omega_5^2)\right] = \ell^2 \tilde{G}_{MN} dx^M dx^N; \quad \tilde{H}(z) = 1 + \frac{\ell^4}{z^4} \tag{5.7}$$

The Polyakov action for this metric is

$$S_{\text{Pol}} = \frac{\ell^2}{4\pi\alpha'}\int d^2\sigma \sqrt{-h} h^{ab}\partial_a X^M \partial_b X^N \tilde{G}_{MN} \tag{5.8}$$

Then:

$$\frac{\ell^2}{4\pi\alpha'} = \sqrt{\frac{\lambda}{4\pi}}; \quad \ell^4 = 4\pi g_s N \alpha'^2 \tag{5.9}$$

For $\alpha' = 0 \Rightarrow \ell = 0$. For any finite z: the metric is $\text{AdS}_5 \times \text{S}^5$: we have zoomed to the near horizon region. But if we started with $z = 0$, then the metric is $\mathbb{R}^{1,9}$.

Conclusion: From the black-brane perspective, taking the limit $\alpha' \to 0$ is equivalent to having SUGRA in $\mathbb{R}^{1,9}$ + String theory on $\text{AdS}_5 \times \text{S}^5$.

As both perspectives have to agree, both conclusions have to agree too. The strong form of the AdS/CFT conjecture goes a bit beyond:

AdS/CFT conjecture (strong form)

$$\mathcal{N} = 4 \text{ SYM} \Longleftrightarrow \text{Type IIB String Theory on } \text{AdS}_5 \times \text{S}^5.$$

But why do we have a string theory after zooming? The energy of the string excitations has $1/\sqrt{\alpha'}$, so why doesn't it blow up, having to dismiss strings at z finite? The key is in the redshift:

$$E_z = \frac{const}{\sqrt{\alpha'}}, \quad E_\infty = E_z\sqrt{g_{00}} = E_z\left(1 + \frac{z^4}{\ell^4}\right)^{-1/4} \approx \frac{\ell}{z}E_z \approx \frac{const}{z} \tag{5.10}$$

An important feature of the holographic duality is then that both sides of the correspondence don't live in the same dimension.

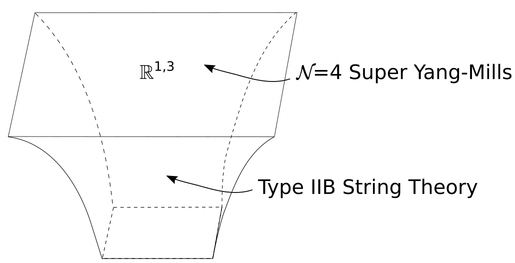


Figure 5.1: Pictorial representation of the conjecture

However, working in string theory is very hard. In order to continue we will need more limits.

't Hooft's limit

As we saw in the previous chapter, 't Hooft's limit involves taking simultaneously $g_s = g_{YM}^2 \to 0, N \to \infty$ so that $\lambda = g_s N$ remains fixed. This limit has different interpretations depending on whether we focus on the boundary or the bulk:

Boundary	Bulk
$\mathcal{N} = 4$ SYM at large N in the 't Hooft's limit	String theory at large N
A diagram: $\lambda^{E-V}N^{V-E+F}$	$g_s \to 0$
Only planar diagrams contribute	"Classical" string theory on AdS_3

Table 5.2: Different perspectives of the 't Hooft limit

Here, by classical string theory we mean that as the genus tends to zero, loop diagrams are suppressed, i.e. there is no genus expansion, just tree level string interactions.

The strong coupling limit

Now that we have fixed limits for g_s, α', something else we can consider is $\lambda \to \infty$.

Boundary	Bulk
Strong coupling	$S_{\text{Pol}} = \sqrt{\frac{\lambda}{4\pi}} \int d^2\sigma \sqrt{h}\, h^{ab} \partial_a X^M \partial_b X^N \tilde{G}_{MN}$
No perturbation theory	$\lambda \to \infty$, string length$\to$0
Mess!	String theory $\to$ SUGRA on $\text{AdS}_5 \times \text{S}^5$

Table 5.3: Different perspectives of the strong coupling limit

This already shows the weak form of the AdS/CFT conjecture:

AdS/CFT conjecture (weak form)

Large N, large λ, $\mathcal{N} = 4$ SYM $\Longleftrightarrow$ Type IIB SUGRA on $\text{AdS}_5 \times \text{S}^5$

Needless to say, the strong form of the conjecture includes the weak one as a limit. Mathematically, the statement of AdS/CFT is then

$$\mathcal{Z}_{\text{CFT}}[J] = \mathcal{Z}_{\text{AdS}}[J] \tag{5.11}$$

5.2 The AdS/CFT dictionary

Here is a summary of the most relevant (known) entries of the duality:

CFT	AdS
$\mathcal{O}(x^\mu)$	$\phi(x^\mu, z)$
$\Delta(\Delta - d)$	$m^2\ell^2$
$T_{\mu\nu}$	$g_{\mu\nu}$
J_μ	A_μ
S_{ent}	$A_{min}/4G_N$
Thermal States	Black Holes
$\cdots_{CFT}$	$\cdots_{AdS}$

Table 5.4: AdS/CFT dictionary

Lecture 6

Applications and examples of the correspondence

We will now attempt to give some examples of tests of the correspondence, where we will find exact matches between the quantities under study. Namely, we will first find the same results for correlation functions of scalar fields, and we will later study black holes.

6.1 Correspondence between 1,2,3-point functions

The (Euclidean) partition function for any theory is

$$\mathcal{Z}[J] = \int D\phi e^{-S_E(\phi)} e^{\int d^dx\sqrt{-g}J(x)\mathcal{O}(x)} \tag{6.1}$$

The n-point functions (correlators) are then computed as

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)...\,\mathcal{O}(x_n)\rangle = \frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}...\frac{\delta}{\delta J(x_n)}\mathcal{Z}[J]\Big|_{J=0} \tag{6.2}$$

In the vacuum of a CFT, 1,2 and 3-point functions are fixed by symmetry:

$$\begin{aligned} &\langle \mathcal{O}(x_1)\rangle = 0, \quad \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{1}{|x_{12}|^{2\Delta}} \\ &\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}} \end{aligned} \tag{6.3}$$

We are going to reproduce the first two with AdS/CFT.

6.1.1 Scalar field action

Consider Euclidean Poincaré AdS, with $\ell_{\text{AdS}} = 1$. We will use coordinates $x^\mu = (t_E, x^i), \mu = 1, ..., d$ for the boundary, and $x^M = (z, x^\mu), M = 1, ..., d+1$ for the bulk. The action for a scalar field in the bulk is then

$$S = -\frac{1}{2}\int d^{d+1}x\sqrt{g}(\partial^M\varphi\partial_M\varphi + m^2\varphi) \tag{6.4}$$

The equations of motion are $(\nabla^2 - m^2)\varphi = 0$, which we can expanded as

$$z^{d+1}\partial_z(z^{-d+1}\partial_z\varphi) + \Box\varphi - m^2\varphi = 0, \quad (\Box = g^{\mu\nu}\partial_\mu\partial_\nu) \tag{6.5}$$

The solutions as $z \to 0$ go like

$$\varphi(z, x^\mu) \sim \begin{cases} \phi_1(x^\mu)z^\Delta + ... \\ \phi_0(x^\mu)z^{d-\Delta} + ... \end{cases} \tag{6.6}$$

A particular solution is the bulk-boundary Green's function:

$$K_\Delta(z, x^\mu; y^\mu) = C_\Delta \left(\frac{z}{z^2 + (x^\mu - y^\mu)^2}\right)^\Delta \tag{6.7}$$

which takes you from (z, x^μ) to y^μ. The complete solution is therefore

$$\varphi(z, x^\mu) = \int_{\text{bdy}} d^d y\, K_\Delta(z, x^\mu; y^\mu)\phi_0(y^\mu) \tag{6.8}$$

It can be proven that

$$\phi_0(x^\mu) = \lim_{z\to 0} z^{\Delta-d}\varphi(z, x^\mu), \quad \phi_1(x^\mu) = C_\Delta \int d^d y \frac{\phi_0(y^\mu)}{(x^\mu - y^\mu)^{2\Delta}} \tag{6.9}$$

Rewriting the action:

$$\begin{aligned} S &= -\frac{1}{2}\int d^d x dz\, z^{-d-1}(z^2\partial_z\varphi\partial_z\varphi + \partial^\mu\varphi\partial_\mu\varphi + m^2\varphi) \\ &= -\frac{1}{2}\int d^d x dz \left[z^{-d+1}\partial_z\varphi\partial_z\varphi - z^{-d-1}(\varphi\partial^\mu\partial_\mu\varphi + m^2\varphi)\right] \\ &= \underbrace{-\frac{1}{2}\int d^d x dz \left[-\partial_z(z^{-d+1}\varphi\partial_z\varphi) - z^{-d-1}(\varphi\partial^\mu\partial_\mu\varphi + m^2\varphi)\right]}_{\text{e.o.m. term}} \underbrace{- \frac{1}{2}\int_{z\to 0} d^d x z^{-d+1}\varphi\partial_z\varphi}_{\text{boundary term}} \end{aligned} \tag{6.10}$$

Assuming that the equations of motion are satisfied, the only term remaining is the boundary one. If we plug in (6.9), we find

$$S_0 = -\frac{1}{2}\int_{z\to 0} d^d x \left[d\,\phi_0(x^\mu)\phi_1(x^\mu) + (d-\Delta)\phi_0^2(x^\mu)z^{d-2\Delta} + ...\right] \tag{6.11}$$

The second term is dangerous (it might blow up). To get rid of it, we need to add counterterms. Specifically,

$$S_{\text{ct}} = \frac{1}{2}(d-\Delta)z^{-d}\int_{\text{bdy}} d^d x\, \varphi^2(z, x^\mu) \tag{6.12}$$

The total action then becomes

$$S_{\text{tot}} = S_0 + S_{\text{ct}} = \frac{(d-2\Delta)}{2}\int d^d x\, \phi_0(x^\mu)\phi_1(x^\mu) \tag{6.13}$$

This is, however, non-local (ϕ_1 contains an integral over position y). We can now check the 1-point function:

$$\langle \mathcal{O}(x_1^\mu) \rangle = \frac{\delta S}{\delta J}\bigg|_{j\to 0} = \frac{\delta S}{\delta \phi_0(x_1^\mu)}\bigg|_{\phi_1=0} = \frac{(d-2\Delta)}{2}\phi_1(x_1^\mu)\bigg|_{\phi_1=0} = 0 \tag{6.14}$$

Not surprisingly, we find the correct expression fixed by symmetry of the CFT. What about the 2-point function?

$$S_{\text{tot}} = \frac{(d-2\Delta)}{2}\int d^dx \int d^dy \, \frac{\phi_0(x^\mu)\phi_0(y^\mu)}{(x^\mu - y^\mu)^{2\Delta}} \Longrightarrow \langle \mathcal{O}(x_1^\mu)\mathcal{O}(x_2^\mu)\rangle = ... = \frac{d-2\Delta}{|x_{12}|^{2\Delta}} \tag{6.15}$$

The correct 3-point function can also be obtained, although we will not check it.

One might think that it is just simpler to check $\mathcal{Z}_{\text{CFT}}[J] = \mathcal{Z}_{\text{AdS}}[J]$, however they are just numbers. This is true for the parition function on S^d or $\mathbb{R}^d$, but it will not be on more complicated manifolds. We will see this now by considering $S^1 \times S^{d-1}$

6.2 Black Holes in $\text{AdS}_3/\text{CFT}_2$

We are going to focus on the $\text{AdS}_3/\text{CFT}_2$ case, as it is the simplest. The CFT side is then $\mathcal{Z}_{\text{CFT}}^{S^1\times S^1 = T^2}$. The AdS side is given by the euclidean path integral (whose classical approximation is

$$\mathcal{Z}_{\text{AdS}}[J=0] = \int [Dg]\, e^{-S_E(g)} \approx e^{-S_{\text{on-shell}}(g_0)} \tag{6.16}$$

where g_0 is a solution to Einstein's equations. Now, the action does have three contributions: the usual Einstein-Hilbert action S_{EH}, the so-called Gibbons-Hawking term (which accounts for the boundary terms from S_{EH}, in order to recover Einstein's equations after integrating by parts on a manifold with boundary), S_{GH}, and the counterterm part S_{ct}:

$$S = S_{\text{EH}} + S_{\text{GH}} + S_{\text{ct}} = -\frac{1}{16\pi G_N}\int_{\mathcal{M}} \sqrt{-g}(R-2\Lambda) - \frac{1}{8\pi G_N}\int_{\partial\mathcal{M}} \sqrt{h}K + \int_{\partial\mathcal{M}} \sqrt{h} f(h) \tag{6.17}$$

This deserves some explanation: our manifold in consideration is $\mathcal{M}$ with metric $g_{\mu\nu}$, which will be bounded in the radial coordinate, $r \in [r_H, r_{max}]$. This boundary is $\partial\mathcal{M}$, with induced metric $h_{\mu\nu}$. K is the extrinsic curvature of the hypersurface $\partial\mathcal{M}$, computed through

$$K = h^{\mu\nu}K_{\mu\nu} = h^{\mu\nu}\nabla_{(\mu}n_{\nu)} \tag{6.18}$$

where n is an inward pointing unit normal vector to $\partial\mathcal{M}$.

Later we will take $r_{max} \to \infty$ (as we are interested in the black hole case, the interesting boundary is r_H), for this reason there will be divergences, regulated with suitable functions $f(h)$.

We will be seeking solutions to the Einstein's equations with the right boundary conditions:

- $J = 0$
- $ds^2_{\text{bdy}} = dt_E^2 + d\varphi^2$ with periodicities $t_E \sim t + \beta, \quad \varphi \sim \varphi + 2\pi$.

Solution 1: Thermal AdS

The first solution we consider is

$$ds^2 = \left(\frac{r^2}{\ell^2} + 1\right) dt_E^2 + \left(\frac{r^2}{\ell^2} + 1\right)^{-1} dr^2 + r^2 d\varphi^2 \tag{6.19}$$

This is basically Figure 6.1. We don't have problems such as closed timelike curves, as it's in Euclidean signature.

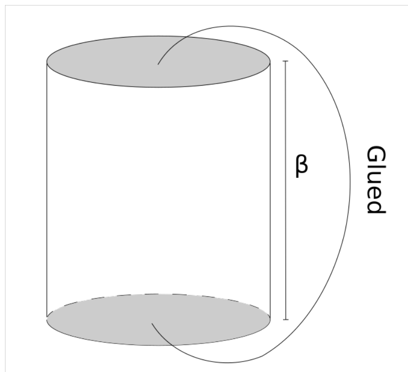


Figure 6.1: Topology of thermal AdS

Solution 2: BTZ black hole

Another solution is

$$ds^2 = \frac{r^2 - r_H^2}{\ell^2} dt_E^2 + \frac{\ell^2 dr^2}{r^2 - r_H^2} + r^2 d\varphi^2 \tag{6.20}$$

To relate it to the usual metric in terms of mass, $8G_N M = r_H^2/\ell^2$. Let's now list the quantities derived from this metric we will be using later on.

- It's three dimensional, therefore $\Lambda = -1/\ell^2$.
- The Ricci scalar is $R = -6/\ell^2$
- The (normalized) inward normal vector at r_{max} is $n_\mu = -\sqrt{\frac{\ell^2}{r_{max}^2 - r_H^2}}\partial_r$
- The induced metric at r_{max} is

$$ds^2 = h_{\mu\nu} dx^\mu dx^\nu = \frac{r_{max}^2 - r_H^2}{\ell^2} dt_E^2 + r_{max}^2 d\varphi^2, \quad \sqrt{h} = r_{max}\sqrt{\frac{r_{max}^2 - r_H^2}{\ell^2}} \tag{6.21}$$

- The extrinsic curvature is found to be $K = \frac{r_H^2 - 2r_{max}^2}{\ell r_{max}}\sqrt{\frac{1}{r_{max}^2 - r_H^2}}$

We can now compute the value of the on-shell action. We compute first the Einstein-Hilbert and Gibbons-Hawking terms:

$$
\begin{aligned}
S_E &= -\frac{1}{16\pi G_N}\int_0^\beta d\tau \int_0^{2\pi} d\varphi \int_{r_H}^{r_{max}} dr\, r\left(-\frac{4}{\ell^2}\right) - \frac{1}{8\pi G_N}\int_0^\beta d\tau \int_0^{2\pi} d\varphi\, r_{max}\frac{r_H^2 - 2r_{max}^2}{\ell^2 r_{max}} \\
&= \frac{2\pi\beta}{4\pi G_N \ell^2}\left(\frac{r_{max}^2}{2} - \frac{r_H^2}{2}\right) - \frac{2\pi\beta}{8\pi G_N}\frac{r_H^2 - 2r_{max}^2}{\ell^2} = \frac{3\beta r_{max}^2}{4G_N\ell^2} - \frac{\beta r_H^2}{2G_N\ell^2}
\end{aligned}
\tag{6.22}
$$

The first term will diverge as $r_{max} \to \infty$, therefore we need a counterterm to annihilate it. But what is $f(h)$? It has to be a diffeomorphism invariant of the boundary metric, i.e.

$$
f(h) = C_1 + C_2 R_h + C_3 R_h^2 + ... \tag{6.23}
$$

where R_h is the Ricci tensor of the boundary metric. It turns out that for this specific case only the first term is needed. Therefore,

$$
S_{\text{ct}} = C_1 \int_{\partial\mathcal{M}} \sqrt{h} = \frac{2\pi\beta C_1}{8\pi G_N}\left(\frac{r_{max}^2}{\ell} - \frac{r_H^2}{\ell} + O(1/r_{max})\right) \tag{6.24}
$$

Imposing that it cancels the divergence implies $C_1 = -3/\ell$. The full action is then

$$
S = S_E + S_{\text{ct}} = -\frac{\beta r_H^2}{8G_N\ell^2} \tag{6.25}
$$

Therefore, by the AdS/CFT duality, $\mathcal{Z}_{\text{CFT}} = \mathcal{Z}_{\text{AdS}} = \exp\left\{\frac{\beta r_H^2}{8G_N\ell^2}\right\}$.

Now, we are going to seek a relation between β and r_H. The easiest way to achieve this is by imposing regularity in the Euclidean signature (this is the standard procedure to bypass the explicit computation carried out by Hawking, much more complicated). We begin with the metric $ds^2 = f(r)dt_E^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{d-1}^2$. Zooming in to the horizon ($r \to x + r_h$) and then doing a change of variables one can see that the polar coordinates metric appears, with $\propto t_E$ as the angular variable, hence the wanted relation can be read off.

This procedure, in our case, yields

$$
\beta = \frac{4\pi\ell^2}{f'(r_H)} = \frac{2\pi\ell^2}{r_H} \Rightarrow r_H = \frac{2\pi\ell^2}{\beta} \tag{6.26}
$$

This, together with a classic result from AdS_3 (Brown & Henneaux, 1986) relating the central charge to the AdS radius, $c = \frac{3\ell}{4G_N}$, allows us to write the previous partition function as

$$
\mathcal{Z}_{\text{CFT}} = \exp\left\{\frac{4\pi^2\beta\ell^4}{8G_N\beta^2\ell^2}\right\} = \exp\left\{\frac{\pi^2\ell^2}{2G_N\beta}\right\} = \exp\left\{\frac{\pi^2\ell c}{3\beta}\right\} \tag{6.27}
$$

The boundary metric of our spacetime is $ds_{\text{bdy}}^2 = dt_E^2/\ell^2 + d\varphi^2$, so we should redefine $\tilde{t}_E = t_E\ell$ such that the Euclidean boundary (where the CFT is defined) is just polar coordinates. This in turn implies a redefinition $\tilde{\beta} = \beta\ell$, hence

$$
\mathcal{Z}_{\text{CFT}} = \mathcal{Z}_{\text{BTZ}} = \exp\left\{\frac{c}{12}\frac{4\pi^2}{\tilde{\beta}}\right\} \tag{6.28}
$$

If we had followed this same procedure with Thermal AdS, we would have instead found

$$\mathcal{Z}_{\text{TAdS}} = \exp\left\{\frac{c}{12}\tilde{\beta}\right\} \tag{6.29}$$

There are apparently two valid solutions. But which one minimizes the free energy?

$$\mathcal{F} = -\frac{1}{\tilde{\beta}}\log\mathcal{Z} = \begin{cases} \mathcal{F}_{\text{BTZ}} & = -\dfrac{c}{12}\dfrac{4\pi^2}{\tilde{\beta}^2} \\ \mathcal{F}_{\text{TAdS}} & = -\dfrac{c}{12} \end{cases} \tag{6.30}$$

We can see there's a phase transition at $\tilde{\beta} = 2\pi$. This is called the Hawking-Page phase transition:

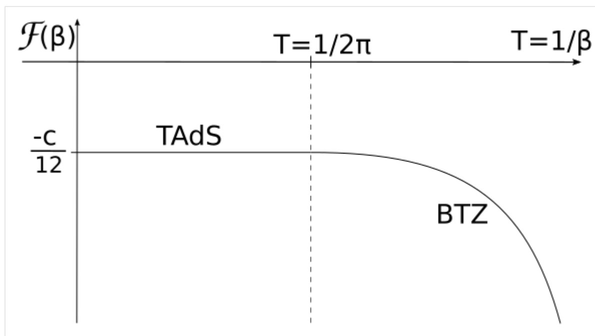


Figure 6.2: Hawking-Page phase transition from Thermal AdS to BTZ.

It is a first order phase transition (the first derivative is already discontinuous). The entropy is found to be

$$S = (1 - \beta\partial_\beta)\log\mathcal{Z} = \begin{cases} S_{\text{BTZ}} & = \dfrac{c}{6}\dfrac{4\pi^2}{\tilde{\beta}} \\ S_{\text{TAdS}} & = 0 \end{cases} \tag{6.31}$$

If we check the usual Bekenstein-Hawking formula for the entropy of a black hole, we find an exact match!

$$S = \frac{A}{4G_N} = \frac{2\pi r_H}{4G_N} = \frac{4\pi^2}{\beta}\frac{\ell}{4G_N} = \frac{c}{6}\frac{4\pi^2}{\beta} \tag{6.32}$$

The energy of each solution is

$$\langle E\rangle = -\partial_\beta \log\mathcal{Z} = \begin{cases} E_{\text{BTZ}} & = \dfrac{c}{12}\dfrac{4\pi^2}{\tilde{\beta}} \\ E_{\text{TAdS}} & = -\dfrac{c}{12} \end{cases} \tag{6.33}$$

The entropy can be rewritten as a function of the energy through the Cardy formula:

$$S(E) = 2\pi\sqrt{\frac{c}{3}E} \tag{6.34}$$

Not long before AdS/CFT was formulated, Strominger & Vafa studied a configuration of N_1 D1-branes and N_2 D5-branes corresponding to extremal black holes in 5d. They found the exact number of microstates of a black hole through a seemingly string-theoretical calculation. But it turns out that they were using, without knowing it, AdS/CFT (as the black hole has a near-horizon AdS geometry).

6.2.1 Modular transformations and 2d CFT's

The partition function for a quantum system, at temperature $T = 1/\beta$ is given by

$$\mathcal{Z}(\beta) = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\, q^{L_0 - \frac{c}{24}}\, \bar{q}^{\bar{L}_0 - \frac{c}{24}} \tag{6.35}$$

If we consider the conformal boundary of thermal AdS_3, as we saw in the first chapter it is topologically a cylinder, with height β as we are in the thermal case. The partition function for this geometry, due to the trace, corresponds to an identification in the time direction, hence what the topology we are dealing with is that of a torus. From the point of view of the 2d CFT, this is viewed as the complex plane t_E, φ with identifications $t_E \sim t_E + \beta$, $\varphi \sim \varphi + 2\pi$. However, notice that we find the exact same torus if we consider the identifications $t_E \sim t_E + 2\pi$, $\varphi \sim \varphi + \beta$.

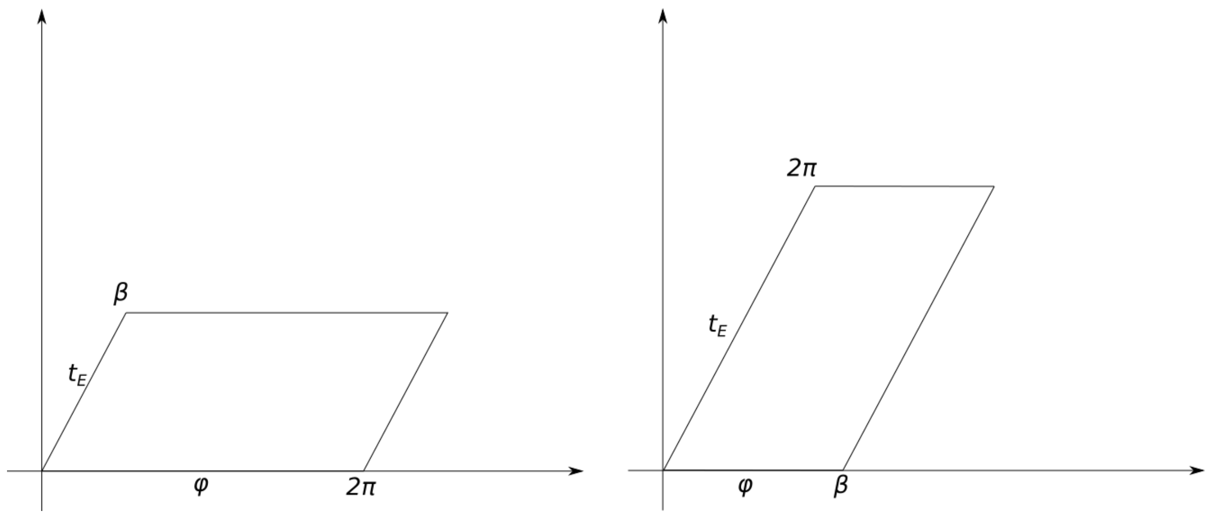


Figure 6.3: Fundamental domain in the complex plane, where the sides have been identified as $t_E \sim t_E + \beta$, $\varphi \sim \varphi + 2\pi$ (left) and $t_E \sim t_E + 2\pi$, $\varphi \sim \varphi + \beta$ (right)

Notice that we can now rescale the whole complex plane defined by those coordinates, in particular we will multiply each axis by $2\pi/\beta$. This implies that we're back to the initial situation (identification) in the φ axis, but the t_E axis is now identified as $t_E \sim t_E + 4\pi^2/\beta$. But as we said, the two tori were the same, and both axis were rescaled in the same fashion, hence we have

$$\mathcal{Z}(\beta) = \mathcal{Z}\left(\frac{4\pi^2}{\beta}\right) \tag{6.36}$$

This phenomenon is called modular invariance, and it is the reason for the behaviour previously found for Thermal AdS and the BTZ black hole. They are then called modular images of each other, as indeed their boundary topologies are the same, the only difference is the change time $\leftrightarrow$ space.

This allows us to relate two different situations, low and high temperature:

1. Low temperature, $\beta >> 1$

$$\mathcal{Z}(\beta) = e^{\frac{c}{12}\beta}\left(1 + \rho(\Delta_1)e^{-\beta\Delta_1} + ...\right) \approx e^{\frac{c}{12}\beta} \tag{6.37}$$

2. High temperature, $\beta << 1$

$$\mathcal{Z}(\beta) = \mathcal{Z}\left(\frac{4\pi^2}{\beta}\right) = e^{\frac{c}{12}\frac{4\pi^2}{\beta}}\left(1 + \rho(\Delta_1)e^{-\beta\Delta_1} + ...\right) \approx e^{\frac{c}{12}\frac{4\pi^2}{\beta}} \tag{6.38}$$

Now, let's focus on the $\beta \to 0$ case. The density of states at a given energy is

$$\rho(E) = \int d\beta\, e^{\beta E}\, \mathcal{Z}(\beta) = \int d\beta\, e^{\beta E}\, e^{\frac{4\pi^2}{\beta}\frac{c}{12}} \tag{6.39}$$

In the saddle point approximation, we are interested in

$$\frac{\partial}{\partial\beta}\left(\beta E + \frac{4\pi^2}{\beta}\frac{c}{12}\right) = 0 \tag{6.40}$$

which is solved by $\beta^* = \sqrt{\frac{4\pi^2 c}{12E}}$, hence

$$\rho(E) \approx e^{\beta^* E}\, e^{\frac{4\pi^2}{\beta^*}\frac{c}{12}} = e^{2\pi\sqrt{\frac{c}{3}E}} \tag{6.41}$$

From statistical physics we know that the entropy is obtained as the logarithm of the number of microstates,

$$S(E) = 2\pi\sqrt{\frac{c}{3}E} \tag{6.42}$$

We have therefore found the acclaimed Cardy Formula! In 1998 Strominger found, using AdS/CFT the exact agreement between this and the entropy of the BTZ black hole.

6.3 Exercises (Lectures 5 & 6)

For these exercises, we set $\ell = 1$.

Problem 1: The constant C_Δ

We saw in class that a solution to the Klein-Gordon equation with arbitrary boundary conditions for the source term $\phi_0(y^\mu)$ is

$$\phi(z, x^\mu) = \int_{\text{bdry}} d^d y K_\Delta(x, x^\mu, y^\mu)\phi_0(y^\mu)\,,$$

where the bulk to boundary propagator K is given by

$$K_\Delta(x, x^\mu, y^\mu) = C_\Delta \left(\frac{z}{z^2 + (x^\mu - y^\mu)^2}\right)^\Delta\,.$$

The goal of this exercise is to show that

$$C_\Delta = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta - d/2)}\,.$$

To do this, we will match the source term. We will demand

$$\lim_{z\to 0} z^{\Delta - d}\phi(z, x^\mu) = \phi_0(x^\mu)\,.$$

Since it is only a normalization, you should do this for constant sources

$$\phi_0(x^\mu) = \phi_0$$

To obtain C_Δ, do the y integrals in the expression for ϕ. Mathematica will be your friend.

Hint: do the integral in polar coordinates.

Problem 2: Counter-terms for scalars

We saw in class that the action for the scalar field is

$$S_0 = -\frac{1}{2}\int_{\text{bdry}} d^d x \left(d\phi_0(x^\mu)\phi_1(x^\mu) + (d - \Delta)\phi_0^2(x^\mu) z^{d-2\Delta} + \dots\right)\,.$$

The second term can be divergent and must be regularized by a counter-term. Try the counter-term

$$S_{\text{ct}} = \alpha z^{-d}\int_{\text{bdry}} d^d x \phi^2(z, x)\,,$$

and determine the value of α such that the problematic term disappears. Write down the remaining action $S_0 + S_{\text{ct}}$.

Imagine you wanted to get rid of other divergences hiding in the $\dots$, what type of other local counter-terms could you add?

Problem 3: Black Hole Thermodynamics

Consider the (Euclidean) five-dimensional AdS-Schwarzschild black hole

$$ds^2 = f(r)dt_E^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2, \qquad f(r) = 1 + r^2 - \frac{M}{r^2}$$

Repeat the process done in class for the BTZ black hole for the black hole above. Compute its on-shell action using the Gibbons-Hawking term as well as counter-terms. The counter-term story is more complicated in five dimensions. You need to add

$$S_{\text{ct}} = \alpha \int_{r=r_{\max}} d^x \sqrt{\gamma} + \beta I_2 \,.$$

Figure out what I_2 is. Remember it has to be made by local data on the boundary, namely it has to be made from the induced metric. It also has to be a scalar. By tuning α and β, cancel all divergences for the action (Do not forget the cosmological constant in the action!!). Compute the free energy. Also compute the entropy using

$$S = (1 - \beta\partial_\beta) \log Z \,,$$

and check that it matches the Bekenstein-Hawking formula.

Proceed in a similar fashion for Thermal AdS_5 with metric

$$ds^2 = (r^2+1)dt_E^2 + \frac{dr^2}{r^2+1} + r^2 d\Omega_3^2$$

Compare the two free energies and determines the value of the temperature where the Hawking-Page phase transition occurs.

Hint: Throughout this problem, you will need to find the horizon radius r_h, where $f(r)$ vanishes. This is related to the temperature through a formula seen in class, but there are multiple branches of solutions. Always consider the larger black holes, the small ones are interesting as well but not the topic of this class, they are much more confusing.

Lecture 7

Renormalization group flows and Holography

In this chapter we will study the holographic meaning of the renormalization group flows from the CFT. Essentially we want to understand the role of the bulk's radial coordinate in the boundary CFT, and this will be linked to the length scale of some QFT.

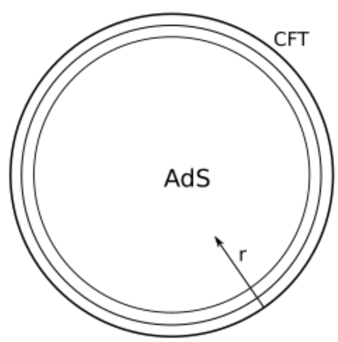


Figure 7.1: In what follows we will explore the foliation of AdS in QFT's, one at each value of r.

In order to perform this study we will begin with a recap of important notions in the renormalization scheme, followed by the holographic view of the flow, and finally we will show it explicitly with an example, a mass deformation of $\mathcal{N} = 4$ SYM.

7.1 Renormalization group in QFT

A useful way to think about particular QFT's is as points in the space of all possible QFT's, an infinite dimensional (finite in the case of renormalizable theories) space that can be parametrized by the values of the coupling constants appearing in the Lagrangian. For example:

$$\mathcal{L} = -\frac{1}{4g_{YM}^2} \operatorname{Tr} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + (\text{other fields}) + (\text{interactions}) \tag{7.1}$$

This is just a special case of

$$\mathcal{L} = \sum_I \lambda_I \mathcal{O}^I \tag{7.2}$$

where $\lambda_I = \{g_{YM}, ...\}$ are the couplings and $\mathcal{O}^I = \{\text{Tr}\, F^2, (\partial\phi)^2, ...\}$ are the operators. A picture of $\mathcal{M}_{QFT}$, the space of QFT's can be done using "coordinates" $\{\lambda_I\}$.

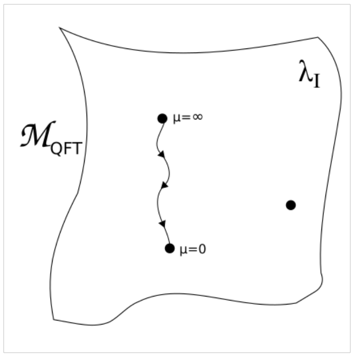


Figure 7.2: Theories in QFT space.

At the quantum level, we know that the coupling constants can run (flow). This flow is characterized by the β-functions: for every λ_I, $\beta_I \equiv \mu\, \partial_\mu \lambda_I$, where μ is the scale of the RG-flow (a parameter of the curve $\lambda_I(\mu)$ along the flow), and β_I is a vector field on $\mathcal{M}_{QFT}$, which can be thought of as a geometric flow on the space of QFT's.

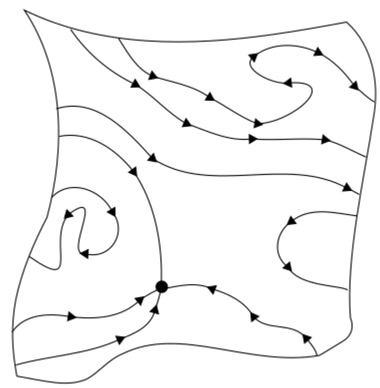


Figure 7.3: Different RG-flows on the QFT space.

The "sinks" of the flow (also called fixed points) are the points in this space where the beta function vanishes; $\beta_I = 0$, and correspond to CFT's, as nothing depends on the scale μ. For instance, in CFT's $T^\mu{}_\mu = 0$, and one can show $T^\mu{}_\mu \sim \beta^I \lambda_I$.

We can study QFT's as deformations of CFT's:

$$\mathcal{L}_{\text{QFT}} = \mathcal{L}_{\text{CFT}} + \sum_I \lambda_I^{def} \mathcal{O}^I \tag{7.3}$$

A simple example of this is a mass deformation, $1/2m^2\phi^2$, which breaks conformal invariance, triggering a flow.

Depending on their scaling dimension, there are three "types" of operators (equivalently, of coupling constants). It's important to notice that they are exchanged in the IR:

$$\lambda_I : \begin{cases} [\mathcal{O}] < d, & \text{relevant} \\ [\mathcal{O}] = d, & \text{marginal} \\ [\mathcal{O}] > d, & \text{irrelevant} \end{cases}$$

where d is the dimension of the spacetime. It is easy to then see that the mass deformation example is an relevant operator, as $[\mathcal{O}] = 1 < 4$, meaning that m^2 becomes increasingly important at low energies. From the point of view of the space of QFT's, relevant deformations bring the theory back to a CFT, whereas irrelevant deformations move the theory away from it. Marginal deformations do neither of these, and define "conformal manifolds".

It is generally believed that RG-flows are not reversible, because they integrate out degrees of freedom. In two dimensions, it can be proven rigorously. This implies that (i.e.) two different, inequivalent initial points in QFT space might flow to the same CFT, hence we can not reverse it. Those two QFT's would then be said to fall in the same universality class.

c-Theorems

The above is summarized in the so-called "c-theorems". Some important examples are

1. Two dimensions: recall that in 2d CFT, the trace anomaly reads $T^{\mu}{}_{\mu} = -\frac{c}{12}R$.

> Zamolodchikov's c-theorem:
>
> If there exists a 2d CFT$_{IR}$ and a 2d CFT$_{UV}$, connected by an RG-flow, then $c_{IR} < c_{UV}$. Hence, c is a function that counts degrees of freedom.

2. Four dimensions: in 4d CFT, the trace anomaly has two terms: $T^{\mu}{}_{\mu} = aE_4 - cW^2_{\mu\nu\rho\sigma}$, where a is in 4d the central charge (it is misleading, indeed), E_4 is the Euler characteristic of the manifold $(\mathcal{M}_4, g)$, c is another anomaly coefficient, and $W_{\mu\nu\rho\sigma}$ is the Weyl tensor.

> Cardy $\oplus$ Komargodski-Schwimmer's a-theorem:
>
> If there exists a 4d CFT$_{IR}$ and a 4d CFT$_{UV}$, connected by an RG-flow, then $a_{IR} < a_{UV}$. Hence, a is a function that counts degrees of freedom.

3. The case for odd dimensions: Nevertheless, for three dimensions it was proven by Casini, Huerta and Myers that there is an F-theorem, analogous to the ones above, with the free energy of the theory on S^3, $F_{S^3} = -\frac{1}{\beta}\log \mathcal{Z}_{S^3}$ being the degree of freedom counting function.

A very well studied example of RG-flow in 4d, which we will derive from holography in the next section, is a mass deformation $\mathcal{N} = 4$ SYM. This model "breaks 3/4 of supersymmetry", and $a_{IR} = \frac{27}{32}a_{UV}$.

7.2 Holographic view of RG-flows

Back to the subject under study: what is the holographic dual of an RG-flow? The claim is, if μ is the renormalization scale between two CFT's, then it is dual to $1/z$, with z the Poincaré radial coordinate.

Recall that $z = 0$ corresponds to the boundary of AdS, and $z = \infty$ corresponds to far in the interior. If we consider CFT$_d$, a scaling corresponds to $\vec{x} \to a\vec{x}, t \to at$, hence $E \to \frac{1}{a}E$. This along with ds^2_{AdS} being invariant under $z \to az$, is a mild justification of why $\mu \to \frac{1}{a}\mu$ has the same scaling as $1/z$.

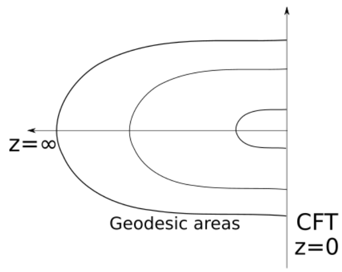


Figure 7.4: Pictorial argument for the scaling

The more information we need to know about the CFT, the more we need to know about the AdS space.

We can also see that each slice of AdS defines a QFT at scale μ.

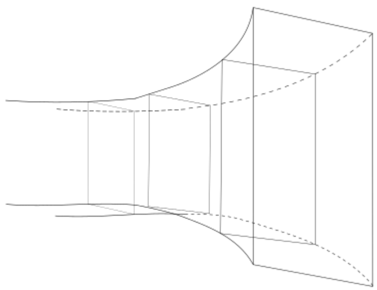


Figure 7.5: Slices of the RG-flow

From the point of view of the CFT, the spacetime slices are generated just as slices of the RG-flow.

In the case of AdS/CFT, as AdS is conformally invariant, the flow is trivial, in the sense that it stays in the same place, meaning that all the QFT slices are just $\mathcal{N}=4$ SYM. As we reviewed in the first chapter, we want to allow for deformations of the AdS interior, which in the context of the holographic renormalization translates into a flow from two AdS regions through a not-exactly-AdS one.

But, how do we trigger flows in AdS/CFT? We need to consider the setting:

$$\text{Supergravity on } \mathrm{AdS}_5 \times S^5 \Leftrightarrow \mathcal{N}=4 \text{ SYM within planar and 't Hooft limit}$$

Upon reducing on S^5, one gets a bunch of scalar fields from (g_{MN}, b_{MN}, Φ), $M, N = 0, ..., 9$. This is called $5d$ $\mathcal{N}=8$. It is a mess, so we are going to study a toy model:

$$S = \int d^{d+1}x\sqrt{g}\,(R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2V(\phi)) \tag{7.4}$$

(this is basically keeping one scalar out of the 42 given by the above configuration). $V(\phi)$ is some potential with many stationary points $\{\phi_k\}$:

$$\left.\frac{\partial V}{\partial\phi}\right|_{\phi_k} = 0 \tag{7.5}$$

Consider expanding it around a critical point:

$$V(\phi_k) = const + \frac{1}{2}m^2\phi_k^2 + \frac{1}{3!}b\phi_k^3 + ... \tag{7.6}$$

where we set $const = \Lambda = -d(d-1)/\ell^2$, as we want the case of $\phi_k = 0$ to correspond to pure AdS. We must respect the BF bound, $-d^2/4 < m^2\ell^2 < 0$. The stationary point is a local maximum for this range. According to AdS/CFT, scalars with masses in this range are dual to operators $\mathcal{O}_\Delta$, with dimension

$$\Delta = \frac{1}{2}(d + \sqrt{d^2 + m^2\ell^2}) \leq d \tag{7.7}$$

therefore we discover that $\mathcal{O}_\Delta$ is a relevant operator! (or at most, marginal). Now recall that a scalar in AdS is given by

$$\phi(x,z) = z^{d-\Delta}(\phi_0(x) + ...) + z^\Delta(\phi_{2\Delta-d}(x) + ...) \tag{7.8}$$

Indeed, plugging in the expression for $\phi(x,z)$ into S_G, and performing the holographic renormalization, the Lagrangian is

$$\mathcal{L}_{\text{QFT}} = \mathcal{L}_{\text{CFT}} + \phi_0 \mathcal{O}_\Delta \tag{7.9}$$

where $\phi_0 \mathcal{O}_\Delta$ is therefore a relevant deformation. This already shows us that RG-flows are triggered in AdS/CFT by introducing scalar fields in AdS with non-normalizable (ϕ_0) modes.

Let us continue. We are interested in asymptotically AdS solutions with scalars turned on. We will use the following ansatz:

$$ds^2 = e^{2A(r)}\delta_{ij}dx^i dx^j + dr^2, \quad \phi = \phi(r) \tag{7.10}$$

where $r = -\ell \log(z/\ell)$. Assuming that V has at least two critical points, $V(\phi_k) = -d(d-1)/\ell_k^2 < 0$,

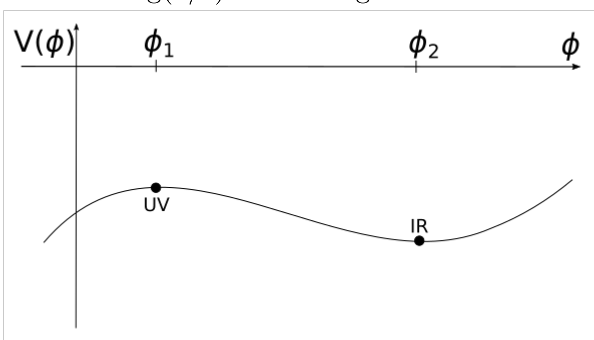


Figure 7.6: Critical points of the assumed potential

As we can see, there are two critical points, the UV one with $m^2\ell_1^2 < 0$, and the IR one with $m^2\ell_2^2 < 0$. The two critical points corresponds to two AdS spaces, with different AdS radii

Plugging the ansatz for ϕ and the metric, the equations of motion imply become

$$(A')^2 = \frac{1}{d(d-1)}[(\phi')^2 - 2V(\phi)], \quad \phi'' + A'\phi' d = \frac{\partial V}{\partial \phi}, \quad A'' = -\frac{(\phi')^2}{d-1} < 0 \tag{7.11}$$

Around a critical point $\phi = \phi_k$, $A(r) = \frac{r}{\ell_k}$ (where ℓ_k is the value of ℓ at ϕ_k) satisfies the equations of motion. Plugging it back in the metric ansatz,

$$ds^2 \approx e^{r^2/\ell_k^2} d\vec{x}^2 + dr^2 = \frac{\ell_k^2}{z^2}(d\vec{x}^2 + dz^2) \tag{7.12}$$

which is indeed AdS, as we expected! The situation, then, is the following:

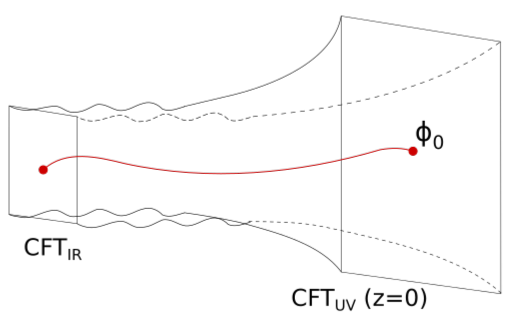


Figure 7.7: Holographic flow from CFT's

ϕ_0 sources an operator $\mathcal{O}_\Delta$, which implies that along the RG-flow we don't find $\mathcal{N} = 4$ SYM, as it is not "perfect" AdS anymore.

Away from AdS, we can still define

$$a(r) = \frac{\pi}{8G_5}\frac{1}{A'(r)^3} \Rightarrow a'(r) = -\frac{\pi}{8G_5}\frac{3A''(r)}{A'(r)^4} > 0 \tag{7.13}$$

monotonically decreasing in z! We have proven the a-theorem with holography.

7.3 Mass deformation of $\mathcal{N} = 4$ SYM: Example

We have played enough with the toy model. The real one does contain $A_\mu, \chi_\alpha, \Phi_{[IJ]}$, where $\Phi_{[IJ]}$ contains (6 real) 3 complex scalars Φ_1, Φ_2, Φ_3. The deformation we consider is

$$\mathcal{L} = \mathcal{L}_{\mathcal{N}=4} + \frac{1}{2}m^2|\Phi_3|^2 + (\mathcal{N} = 1 \text{ completion}) \tag{7.14}$$

This model breaks the $\mathcal{N} = 4$ conformal invariance, but just so slightly that the $\mathcal{N} = 1$ completion allows to still keep supersymmetry.

To understand the RG-flow triggered by $\mathcal{O}_{\Delta=2} = |\Phi_3|^2$, we need to find supergravity scalars dual to $\mathcal{O}_\Delta$. In this scenario, the dictionary has been found to be

SUGRA scalar	$m^2\ell^2$	$\mathcal{O}$	$\mathrm{SU}(4)_R$	$\Delta = 2 + \sqrt{4 + m^2\ell^2}$
Φ	0	$\mathrm{Tr}\, F \wedge \star F$	1	4
$C_{(0)}$	0	$\mathrm{Tr}\, F \wedge F$	1	4
φ_1	-3	$\mathrm{Tr}\, \lambda_{(a}\lambda_{b)}$	10	3
$\bar{\varphi}_1$	-3	$\mathrm{Tr}\, \bar{\lambda}_{(a}\bar{\lambda}_{b)}$	$\overline{10}$	3
$\binom{\alpha}{\chi}$	-4	$\mathrm{Tr}\, \Phi_i\Phi_j - \frac{1}{6}\delta_{ij}\, \mathrm{Tr}\, \Phi_k\Phi_k$	20	2

Table 7.1: $\mathrm{AdS}_5/\mathrm{CFT}_4$ dictionary

The bosonic part of 5d $\mathcal{N} = 8$ SUGRA (type IIB reduced on S^5) is

$$S = \int d^5x\sqrt{g}\left(R - \frac{1}{2}|\partial_\mu\Phi^I|^2 - 2V(\Phi^I) + ...\right), \quad I = 1, ..., 42 \tag{7.15}$$

Because of SUSY,

$$V(\phi^I) = \frac{4}{\ell^2}\left[\frac{1}{2}\left(\frac{\partial V}{\partial\alpha}\right)^2 + \frac{1}{2}\left(\frac{\partial V}{\partial\chi}\right)^2 + \frac{4}{3}W^2\right] \tag{7.16}$$

where

$$W^2 = \frac{1}{4\rho^2}\left(\cosh(2\chi)(\rho^6 - 2) - (3\rho^6 + 2)\right), \quad \rho = e^{2\alpha} \tag{7.17}$$

The equations of motion can be written as:

$$\rho' = \frac{1}{6\ell^2}\rho^2\frac{\partial W}{\partial\rho}, \quad \chi' = \frac{1}{\ell}\frac{\partial W}{\partial\chi}, \quad A' = -\frac{2}{3\ell}W \tag{7.18}$$

The fixed points of the RG-flow occur at $\frac{\partial V}{\partial\chi} = \frac{\partial V}{\partial\alpha} = 0$, with solution:

a) $\chi = 0$, $\alpha = 0$

b) $\chi = \pm \log 3^{1/2}$, $\alpha = \log 2^{1/6}$

In the fixed points,

$$A'\big|_a = \frac{1}{\ell} \to A\big|_a = e^{r/\ell}, \quad A'\big|_b = \frac{2^{5/3}}{3\ell} = \frac{1}{\tilde{\ell}} \to A\big|_a = e^{r/\tilde{\ell}} \tag{7.19}$$

The central charge in this setting is $a = \frac{\pi \ell^3}{2G_5}$, therefore

$$\frac{a_{UV}}{a_{IR}} = \left(\frac{\ell_{UV}}{\ell_{IR}}\right) = \left(\frac{\ell}{\tilde{\ell}}\right)^3 = \frac{27}{32} < 1 \tag{7.20}$$

which is the result we showed before.

This is a good test of the AdS/CFT conjecture away from the conformal case, as it describes the whole flow holographically. We've seen it for this model but it has also been shown for other ones.

Lecture 8

Other well-established dictionaries of AdS/CFT

8.1 Brane engineering

The basic idea in this section can be summarized as: take D-branes (or M-branes), stack them, and backreact on the geometry (Figure 8.1).

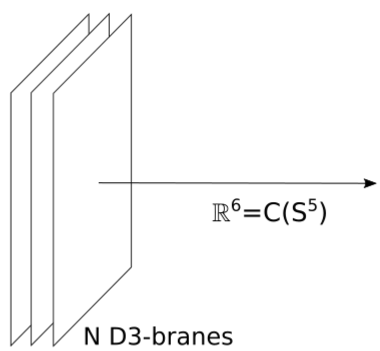


Figure 8.1: Stack of N D3-branes, with transversal $\mathbb{R}^6$ directions.

What is the physical interpretation of this setting? It is well known that one can engineer quantum field theories by suitably stacking branes on top of each other. There will be some spatial directions not filled with them, which will provide us with extra degrees of freedom.

With this in mind, let's now recall $\mathcal{N} = 4$ Super Yang-Mills: it was a 4d quantum field theory (which can be realized with the use of D3-branes), with 6 scalar degrees of freedom (corresponding to through transversal spatial dimensions), which can be internally rotated through the SO(6) R-symmetry (corresponding to the SO(6) rotational symmetry of the transverse space), etc...

Now, the ambient space is 10-dimensional (or 11 if within M-theory framework) flat spacetime, therefore the transverse directions have to be locally flat too. One way to view this is as a conical geometry, which is locally flat too.

Conical geometries: If one has a manifold $\mathcal{M}_d$, a cone can be defined as $C(\mathcal{M}_d) = \mathbb{R}_+ \times \mathcal{M}_d$. The local geometry is

$$ds^2_{C(\mathcal{M})} = dr^2 + r^2 ds^2_{\mathcal{M}} \tag{8.1}$$

For example, polar coordinates in the plane can be seen as a cone of S^1.

Upon backreaction, the near-horizon region of the branes looks like

$$ds^2_{10d} = ds^2_{\mathrm{AdS}_5} + ds^2_{\mathcal{M}_5} \tag{8.2}$$

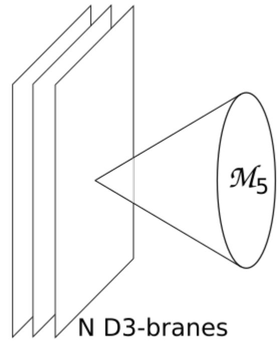


Figure 8.2: Stack of branes backreacting

To have at least $\mathcal{N} = 1$ SUSY in 4d, the manifold $\mathcal{M}_5$ needs to be a special kind of manifold: Sasaki-Einstein $\Leftrightarrow C(\mathcal{M}_5)$ is a Calabi-Yau threefold.

Einstein manifolds: A manifold is said to be of Einstein type if its Ricci tensor is proportional to the metric tensor,

$$R_{\mu\nu} = k\, g_{\mu\nu} \tag{8.3}$$

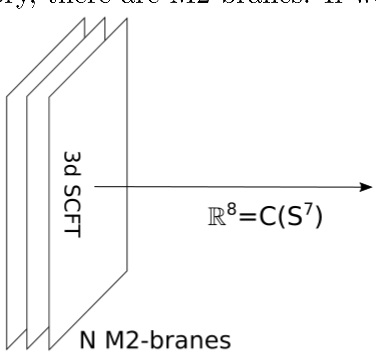
where k is the proportionality constant. They are called like that because they are solutions to the vacuum field equations.

Calabi-Yau manifolds: A complex manifold is said to be of Calabi-Yau type if it is a compact Kähler manifold (meaning that it has complex structure, Riemannian structure and symplectic structure equipped), with vanishing first Chern class, and a Ricci-flat metric. The extra dimensions in superstring theory are conjectured to take this form.

Examples: $\mathcal{M}_5 = S^2 \times S^3$??????

——————WTF——————

In M-theory, there are M2-branes. If we make them backreact,



$\Longleftrightarrow \mathrm{AdS}_5 \times S^7$.

In 3d supersymmetry with $\mathcal{N}$ supercharges, the R-symmetry is SU($\mathcal{N}$).

The corresponding field theory was found with AdS/CFT, and it is called ABJM theory [11], which has Chern-Simons $\mathrm{SU}(N)_1 \times \mathrm{SU}(N)_{-1}$ terms. The same game can be played with the M5-branes and what theory does it correspond to?.

Let us summarize the state-of-the-art of this landscape of theories. First the ones "on a solid footing":

CFT_d	Brane/transverse space	SUGRA solution	"tests"
4d $\mathcal{N}=4$ SYM	N D3/$\mathbb{R}^6$	IIB: $\text{AdS}_5 \times S^5$	too many to list
4d $\mathcal{N}=1$ quiver gauge theories	N D3/$C(SE_5)$	IIB: $\text{AdS}_5 \times SE_5$	central charges, RG-flows,...
3d $\mathcal{N}=8$ ABJM CS	N M2/$\mathbb{R}^8$	M: $\text{AdS}_7 \times S^7$	free energies, Wilson loops, anomalous dimensions,...
3d $\mathcal{N}=3$ CS	N M2/$C(SE_7)$	M: $\text{AdS}_4 \times SE_7$	free energies, Wilson loops,...
6d $(2,0)$ A_N	N M5/$\mathbb{R}^5$	M: $\text{AdS}_7 \times S^4$	anomalies
6d $(1,0)$	N M5/$(\mathbb{R}^5/\Gamma)$	M: $\text{AdS}_7 \times (S^4/\Gamma)$	anomalies
5d $\mathcal{N}=1$ $\text{Sp}(N)_{N_f}$	N D4,D8,O_8	IIA: $\text{AdS}_6 \times S^4$	free energies, flows,...

Table 8.1: Correspondences on a solid footing:

8.2 Embedding AdS and black holes

One way to make progress in understanding AdS_2 is by embedding it in a higher-dimensional example of $\text{AdS}_{d+1}/\text{CFT}_d$. For example, if we consider 4d SUGRA

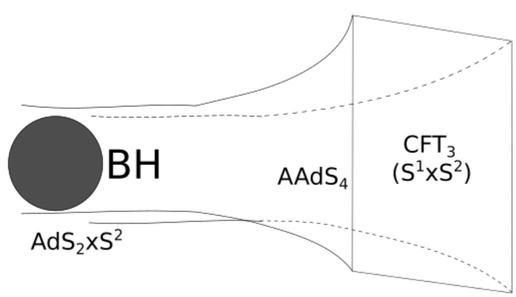


Figure 8.3: Black hole embedded in a higher dimensional AdS

$$\frac{A}{4G_N^{(4)}} = S_{\text{BH}} = \log \Omega = \log \mathcal{Z}_{S^1 \times S^2}^{\text{CFT}_\infty} \tag{8.4}$$

A black hole solution in AdS_4 with $\text{U}(1)^4$ gauged symmetry, $\mathcal{N}=8$ SUSY:

$$ds_4^2 = -e^{K(\Phi)} \left(r - \frac{1}{r}\right)^2 dt^2 + e^{-K(\Phi)} \left(r - \frac{1}{r}\right)^{-2} dr^2 + 2e^{-K(\Phi)} r^2 d\Omega_2^2 \tag{8.5}$$

where $K(\Phi)$ is some function, $\Phi = \Phi(r)$, $A_\mu = A_\mu(r)$. Its entropy is given by

$$S_{\text{BH}} = \frac{A}{4G_N} = -\frac{1}{3} N^{3/2} \sqrt{2\Phi_1(r_1)...\Phi_n(r_n)} \sum_{I=1}^{4} \frac{n_I}{\Phi_I(r_n)} \tag{8.6}$$

In a “completely unrelated” setting...

$$
\begin{aligned}
\mathcal{Z}^{\text{ABJM}}_{S^1\times S^2} = \frac{1}{(N!)^2}\sum_{m,\tilde{m}}\oint\prod_{i=1}^{N}\frac{du_i}{2\pi i u_i}\,\frac{d\tilde{u}_i}{2\pi i\tilde{u}_i}u_i^{m_i}\,\tilde{u}_i^{-\tilde{m}_i}\prod_{i\neq j}\left(1-\frac{u_i}{u_j}\right)\left(1-\frac{\tilde{u}_i}{\tilde{u}_j}\right) \\
\prod_{i=j}\prod_{I=1,2}\left(\frac{u_i}{\tilde{u}_j}e^{i\Phi_I}\right)^{m_i-\tilde{m}_j-n_I+1}\cdot\prod_{I=3,4}\left(\frac{\tilde{u}_i}{u_j}e^{i\Phi_I}\right)^{\tilde{m}_i-m_j-n_I+1}
\end{aligned}
\tag{8.7}
$$

In the large N limit, $\mathcal{Z}^{\text{ABJM}}_{S^1\times S^2}$ will become S_{BH} when taken the logarithm. One can show that for any 3d $\mathcal{N}=2$ SCFT on $S^1\times\Sigma_{g>1}$ exists an AdS_4 black hole with $\text{AdS}_2\times\Sigma_g$ near horizon region.

Bibliography

[1] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Int. J. Theor. Phys. **38** (1999) 1113 [Adv. Theor. Math. Phys. **2** (1998) 231] [hep-th/9711200].

[2] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2** (1998) 253 [hep-th/9802150].

[3] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," Phys. Lett. B **428** (1998) 105 [hep-th/9802109].

[4] H. Nastase, "Introduction to the ADS/CFT Correspondence,"

[5] P. C. West, "Introduction to supersymmetry and supergravity,"

[6] M. Bertolini, "Lectures on Supersymmetry,"

[7] C. Fefferman,C. R. Graham. "Conformal invariants,"in *Élie Cartan et les mathématiques d'aujourd'hui* (Astérisque, 1985), 95-116

[8] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions," Nucl. Phys. B **72** (1974) 461.

[9] M. Banados, C. Teitelboim and J. Zanelli, "The Black hole in three-dimensional space-time," Phys. Rev. Lett. **69** (1992) 1849 [hep-th/9204099].

[10] K. Skenderis, "Lecture notes on holographic renormalization," Class. Quant. Grav. **19** (2002) 5849 [hep-th/0209067].

[11] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, "N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," JHEP **0810** (2008) 091 [arXiv:0806.1218 [hep-th]].